Improved Performance of Reduced M-ary Orthogonal Signaling Using Reed-Solomon Codes

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Abstract- **This paper presents a comparison of communication systems using different signal constellation sizes and Reed-Solomon code sizes with different rates so that the overall required bandwidth is the same for each system. In these comparisons the channel symbol size is smaller than the code symbol size, so that a code symbol contains parts of more than one channel symbols. Thus, the normal assumption of independent code symbols does not apply. Instead consideration must be taken to obtain the best arrangement of channel symbols in each code symbol. Analytical expressions are derived to compare the bit error probability performance of comparable systems based on individual codewords.**

I. Introduction

Normally for error control on a communication system using M-ary orthogonal signaling, an extended Reed-Solomon code of length n, which is equal to the size of the signal constellation, is used. A larger signal constellation requires a more complex receiver system. For example with M-ary orthogonal signaling and noncoherent detection, $n=M$ envelope detectors or $2M$ squaring devices are needed to implement the M-ary receiver. In order to reduce the complexity of the receiver, it is desired to reduce the signal constellation size. However, a larger constellation size for an uncoded system results in a lower probability of bit error than if a smaller signal set is used. Likewise, for a fixed rate code, a larger code size results in a lower probability of bit error. Therefore, to compensate for a loss of performance with a reduced signal constellation size, more powerful codes are needed. A system using a signal constellation of size M with code symbol alphabet size q, where $M<q$, can be found that uses more than one channel symbol to represent a code symbol. Since a smaller signal constellation is used, a gain in bandwidth is obtained. To keep the bandwidth requirements the same for a fair comparison between the two systems, the code rate of the system using the q constellation size is decreased which increases the amount of redundancy used. The added redundancy provides better error control performance for the system.

For example let a type one system be a traditional system using M_1-ary signaling and a (n_1,k_1) extended Reed-Solomon code. For this system the code symbol alphabet size of q_1 equals M_1, so each code symbol is sent as one channel symbol. The length of the code, n_1, for an extended RS code is the same as q_1, and $R_1 = k_1 / n_1$.

Let a type two system use M_2-ary signaling where $M_2 < M_1$. The code symbol alphabet is kept the same, so the code is now a RS (n_1, k_2) code. Since a smaller channel symbol set is used, the code rate of the second code, R_2, is reduced by a factor of M_2/M_1 to compensate for this smaller symbol set. However, $\log_2 M_1 / \log_2 M_2$ channel symbols must be sent for each code symbol, so R_2 must also be increased by this factor for an equivalent bandwidth. The resulting rate is $R_2 = k_2 / n_1$, where

$$k_2 = k_1 \left(\frac{M_2}{M_1} \right) \left(\frac{\log_2 M_1}{\log_2 M_2} \right). \tag{1}$$

Other systems can be found using different signal constellations as long as the resulting rate produces a valid code. For some choices of M_1, k_1, and M_2 a fractional k_2 results, which implies that no RS code of that size exists that can be exactly compared to the original system.

A type three system is also considered if $\log_2 M_1 / \log_2 M_2$ is an integer. This system uses a code alphabet of size M_2 instead of M_1 and uses the same signal constellation size, M_2, as a type two system. To maintain the same bandwidth, the code rate is kept the same. Since a code symbol and a channel symbol are the same for this system, $\log_2 M_1 / \log_2 M_2$ code symbols are transmitted using this system for every one code symbol transmitted with a type two system. Therefore, the code length is given as

$$n_3 = n_1 \left(\frac{\log_2 M_1}{\log_2 M_2} \right), \tag{2}$$

and correspondingly the data length as

$$k_3 = k_1 \left(\frac{M_2}{M_1} \right) \left(\frac{\log_2 M_1}{\log_2 M_2} \right)^2 . \tag{3}$$

Since the code length is no longer equal to the alphabet size, a Reed-Solomon code cannot be used. Another code, such as a non-binary cyclic code, with the desired length and rate is needed.

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II. SYSTEM PERFORMANCE

The performance of different systems is compared where each system has the same energy and bandwidth requirements. The codes used for the systems are extended Reed-Solomon codes where possible, and non-binary BCH codes elsewhere. Errors-only decoding is used, where the minimum distance for the Reed-Solomon codes, and the design distance for the BCH codes determine the number of errors that can be corrected.

Consider a system using M-ary orthogonal signaling where noncoherent demodulation is used. The channel is a non-selective Raleigh fading channel. From [1,2] the probability of receiving a channel symbol in error is

$$p_{ce} = M^{-1}\sum_{j=2}^{M}\binom{M}{j}(-1)^{j}\left[1+(1-j^{-1})E_s/N_o\right]^{-1}, \qquad (4)$$

where E_s/N_0 is the symbol energy to noise ratio. For a fair comparison between systems of different channel symbol alphabets, the symbol energy is converted to bit energy by $E_s = E_b \log_2 M$. Error correcting codes introduce redundancy, which increases the channel symbol transmission rate and reduces the received energy per symbol by the code rate, R, where $R=k/n$.

For systems where the signal constellation size, M, is the same as the code alphabet size, q, one code symbol is sent as a channel symbol. Therefore, the probability of code symbol error is the same as the probability of channel symbol error. If $q>M$ and $\log_2 q/\log_2 M$ is an integer, then a code symbol is transmitted as a series of $\log_2 q/\log_2 M$ channel symbols. Any channel symbol error in the sequence will produce a code symbol error, so the probability of a code symbol error is [3]

$$p_e = 1-(1-p_{ce})^{\log_2 q/\log_2 M}, \qquad (5)$$

assuming the channel symbol errors are independent.

If $\log_2 q/\log_2 M$ is not an integer, then there are a noninteger number of channel symbols in a code symbol, and a code symbol will contain fractional parts of channel symbols. The performance of the system will depend on how the channel symbols are divided within the code symbols. The best transmission pattern of channel symbols, is the pattern that minimizes the probability of code symbol error, and requires the minimum number of channel symbols to transmit a codeword. As shown in Theorem 1, the pattern that satisfies the first criteria is to use whole channel symbols in the code symbol when possible, and when fractions of channel symbols must be used, to keep these fractions as large as possible. For $q=2^m$ and $M=2^b$, if $b/(m-b)$ is an integer then this pattern also keeps the number of channel symbols to a minimum. If $b/(m-b)$ is not an integer, then some code symbols may contain fractions of two different channel symbols to minimize the total number of channel symbols.

Theorem 1 The optimum transmission pattern for a $q=2^m$-ary code symbol using $M=2^b$-ary channel symbols, $b<m$, is to use the largest number of channel symbols along with the remaining equivalent bits from a single channel symbol.

Proof: Assume $m<2b$. If $m\geq 2b$ each code symbol will contain a number of complete channel symbols and the same results hold. Let p_{e_i} be the probability of a code symbol error given that the code symbol contains i bits from one channel symbol, and m-i bits from another channel symbol. This probability is given by

$$p_{e_i} = 1-(1-p_{ce_i})(1-p_{ce_{m-i}}), \qquad (6)$$

where p_{ce_i} is the probability that at least one of the first i bits of an b bit channel symbol is in error. This probability is given by [4]

$$p_{ce_i} = \frac{2^b - 2^{b-i}}{2^b - 1}\, p_{ce}, \qquad (7)$$

where again p_{ce} is the probability that a channel symbol is received in error. By inspection $p_{e_i} = p_{e_{m-i}}$ and only the case where $i \geq (m+1)/2$ must be considered.

Lemma 1 $p_{e_i} \leq p_{e_{i-1}}$

Proof:

$$\begin{aligned}
p_{e_i} - p_{e_{i-1}} &= 1-(1-p_{ce_i})(1-p_{ce_{m-i}})-1 \\
&\quad +(1-p_{ce_{i-1}})(1-p_{ce_{m-i+1}}) \\
&= p_{ce_i} + p_{ce_{m-i}} - p_{ce_{i-1}} - p_{ce_{m-i+1}} - p_{ce_i}p_{ce_{m-i}} \\
&\quad + p_{ce_{i-1}}p_{ce_{m-i+1}} \\
&= \frac{-2^{b-i} - 2^{b-m+i} + 2^{b-i+1} + 2^{b-m+i-1}}{2^b - 1}\, p_{ce} \\
&\quad + \frac{2^{2b-m+i} + 2^{2b-i} - 2^{2b-m+i-1} - 2^{2b-i+1}}{(2^b-1)^2}\, p_{ce}^2 \\
&= \frac{-2^{b-i} + 2^{b-m+i-1}}{2^b - 1}\, p_{ce}\left[\frac{2^b}{(2^b-1)}p_{ce} - 1\right] \leq 0,
\end{aligned}$$

since $p_{ce} \leq \frac{2^b - 1}{2^b}$.

Lemma 1 shows that whole channel symbols should be used in the makeup of code symbols.

Lemma 2

$$1-(1-p_{ce_i})(1-p_{ce_{m-i}}) \leq 1-(1-p_{ce_i})(1-p_{ce_j})(1-p_{ce_{m-i-j}}).$$

Proof: Equivalently, prove $(1-p_{ce_i}) \geq (1-p_{ce_{i-j}})(1-p_{ce_j})$ (replace m-i by i).

$$(1-p_{ce_{i-j}})(1-p_{ce_j})-(1-p_{ce_i})$$
$$=\left(1-\frac{2^b-2^{b-i+j}}{2^b-1}p_{ce}\right)\left(1-\frac{2^b-2^{b-j}}{2^b-1}p_{ce}\right)-\left(1-\frac{2^b-2^{b-i}}{2^b-1}p_{ce}\right)$$
$$=\frac{\left(2^b-2^{b-i+j}\right)\left(2^b-2^{b-j}\right)}{\left(2^b-1\right)^2}p_{ce}{}^2-\frac{2^b-2^{b-i+j}-2^{b-j}+2^{b-i}}{2^b-1}p_{ce}$$
$$=\frac{\left(2^b-2^{b-i+j}\right)\left(2^b-2^{b-j}\right)}{2^b\left(2^b-1\right)}p_{ce}\left[\frac{2^b p_{ce}}{2^b-1}-1\right]\le 0.$$

Lemma 2 shows that when part of a channel symbol is used to complete a code symbol it should be kept as large as possible. Theorem 1 follows.

Assume $m<2b$. If $m\geq 2b$, then each codeword will contain multiple whole channel symbols and the following probabilities of error can be adjusted accordingly. If $b/(m-b)$ is an integer, then each code symbol in a codeword will have the same arrangement of channel symbols: a code symbol will contain one channel symbol and m-b bits from an additional channel symbol. The probability of code symbol error for this case is

$$p_e=1-(1-p_{ce})\left(1-\frac{2^b-2^{b-i}}{2^b-1}p_{ce}\right). \tag{8}$$

The probability of a codeword error is the probability that more than t code symbols were in error,

$$P_w=\sum_{i=t+1}^{n}\binom{n}{i}p_e^i(1-p_e)^{n-i}. \tag{9}$$

If $b/(m-b)$ is not an integer, the probability of code symbol error will not be the same for each code symbol in the codeword, and will depend on the channel symbol arrangement. For example if a 32-ary RS code and 8-ary signaling is used then m=5 and b=3. Each code symbol will contain one complete channel symbol and 2 bits from other channel symbols. Two-thirds of the code symbols will have the remaining 2 bits from one channel symbol, while one-third of the code symbols will have the remaining 2 bits from each of two different channel symbols. This arrangement is necessary to ensure that the minimum number of channel symbols is used to transmit a code word. Since the code symbols are different the probability of code symbol error will differ for the two different arrangements. Let $p_{e'_2}$ be the probability of error for the code symbol that has the remaining 2 bits from one channel symbol, and $p_{e'_1}$ be the probability of error for the code symbol that has the remaining 2 bits from two different channel symbols. These probabilities are given as [4]

$$p_{e'_2}=1-(1-p_{ce})\left(1-\frac{6}{7}p_{ce}\right) \tag{10}$$

and

$$p_{e'_1}=1-(1-p_{ce})\left(1-\frac{4}{7}p_{ce}\right)^2. \tag{11}$$

The resulting probability of codeword error is

$$P_{w_{m=5,b=3}}=$$
$$\sum_{j=t+1}^{n}\sum_{i=i^*}^{j^*}\binom{22}{i}p_{e'_2}^i(1-p_{e'_2})^{22-i}\binom{10}{j-i}p_{e'_1}^{j-i}(1-p_{e'_1})^{10-j+i}, \tag{12}$$

where $i^*=\max(0,j-10)$ and $j^*=\min(j,22)$.

The most likely event causing a received word to be decoded in error, is that t+1 of the code symbols are received in error. The receiver decodes this word to the closest codeword by introducing t additional errors for a total of $2t+1=d_{\min}$ errors. Thus, the decoded code symbol error can be approximated as

$$P_e\approx\left(\frac{d_{\min}}{n}\right)P_w. \tag{13}$$

If all code symbol errors are equally likely, then, from [1], the probability of a bit error is

$$P_b=P_e\left(\frac{q}{2(q-1)}\right), \tag{14}$$

where q is the size of the code symbol alphabet.

III. Results

Figure 1 shows the performance of four systems using an 64-ary extended RS code for the first three systems and an 8-ary BCH code for the fourth system. The RS (64, 60) 64-ary signaling is a type one system, while the RS (64, 36) 32-ary signaling and RS (64, 15) 8-ary signaling are type two systems, and the 8-ary BCH (128, 30) 8-ary signaling is a type three system. The rate of each of the codes has been adjusted so the required bandwidth is the same for each system. As seen from Figure 1, the systems with the smaller signal constellations outperform the 64-ary signaling system as the bit energy-to-noise ratio is increased due to the increased error protection from the lower rate codes. Similar performance results can be seen in Figure 2 for three systems that use 32-ary extended RS codes. The results presented here are for individual code symbol decisions as opposed to [5] which are for multiple code symbol decisions or equivalently longer codes.

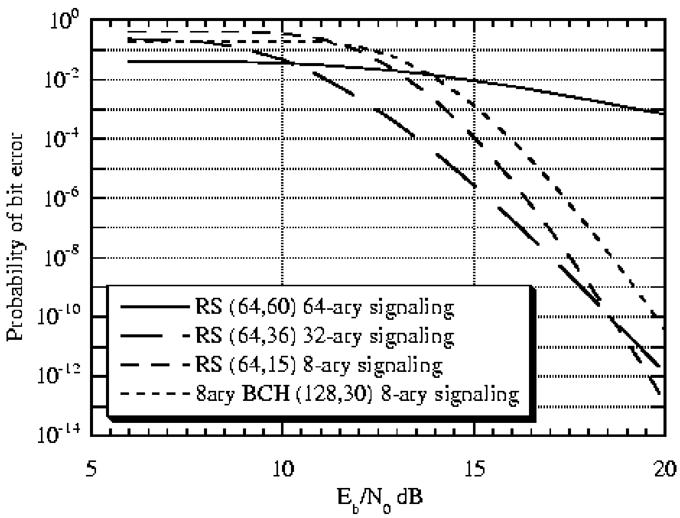

Fig. 1. Systems based on 64-ary Orthogonal Signaling, RS (64,60).

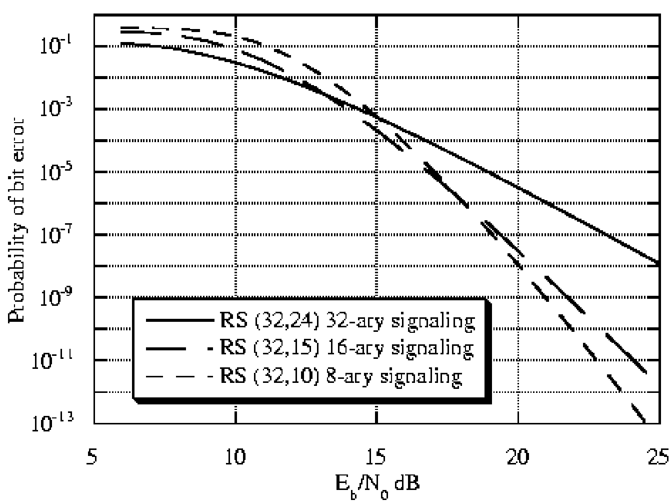

Fig. 2. Systems based on 32-ary Orthogonal Signaling, RS (32,24).

IV. CONCLUSIONS

This paper has presented reduced M-ary orthogonal systems that use error control codes to improve performance. The performance of the systems was developed taking into account the probability of code symbol error, which depends on the arrangement of channel symbol errors within the code symbol. Results show that as the energy-to-noise ratio is increased, the reduced systems outperform the systems with larger signal constellation sizes in terms of bit error probability due to the improved error performance of the RS and BCH codes.

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