POWER SPECTRAL DENSITY OF GMSK MODULATION USING MATRIX METHODS

*Kiran Kuchi**
Motorola Labs, Access Technologies Research, Fort Worth, TX 76137.
email: QKK004@email.mot.com

Vasant K. Prabhu
Department of Electrical Engineering, Box 19016, The University of Texas at Arlington, TX 76019.

ABSTRACT

Gaussian Minimum Shift Keying (GMSK) modulation is widely used in cellular and PCS applications. The popular GSM system uses GMSK. One of the important considerations in the use of modulation is the average power spectral density (psd) and especially the spectral density characteristics on the tails. No method, other than simulation, is currently available to determine the psd of GMSK. In this paper we give closed form expressions for computing the power spectrum using vector-matrix techniques. The psd is expressed in a compact Hermitian form suitable for numerical computation. The psd can be calculated to any desired accuracy by using this method. Results are given for various Gaussian base band filter Bandwidth-Time products and modulation pulse truncation lengths. The formulas given in this paper can be extended to arbitrary pulse shapes and arbitrary modulation indices.

I. INTRODUCTION

To increase the spectral efficiency of mobile radio transmission systems very tight channel spacings are required. Typically the out of band radiated power in the adjacent channels is specified to be 60-80 dB below that of the desired channel. The nonlinear power amplifier in the final stage of the transmitter further limits the choice of a given modulation scheme. In mobile radio transmission, ideally a continuous phase, constant envelope modulation is often desired. Murata and Hirade proposed [1] a premodulation Gaussian filtered minimum shift keying modulation scheme as a solution to meet the given spectral requirements. Excellent spectral characteristics are obtained as a result of inter symbol interference (ISI) generated by the pre-modulation Gaussian filter. The degradation in bit error rate due to ISI is shown to be less than 0.7 dB compared to MSK scheme.

The knowledge of power spectral density is important not only to define the bandwidth requirements but also in evaluating the co-channel and adjacent channel interference effects. In reference [1] the spectrum is calculated using simulation methods. No analytical results have been obtained so far to derive the psd. In this paper we give closed form expressions for calculating the GMSK spectral density using vector-matrix techniques. The psd is expressed in a compact Hermitian form suitable for numerical computation. Results are given for various Gaussian base band filter Bandwidth-Time products and modulation pulse truncation lengths. The method of determining this truncation length is discussed. The formulas given in this paper can be extended to arbitrary pulse shapes and arbitrary modulation indices.

*. This work was done while the author was at the University of Texas at Arlington.

II. TRANSMITTER STRUCTURE

The transmitter structure of the GMSK modulation scheme is given Figure1.

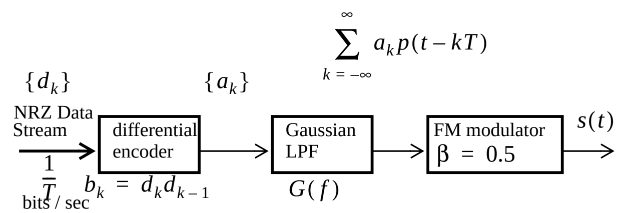

Figure 1. Transmitter.

Differentially encoded rectangular data stream is filtered using a Gaussian low pass filter of 3-dB band width B_t. The impulse response of the gaussian filter is given by

$$G(f) = e^{-\left(\frac{\pi f}{\alpha}\right)^2}, \tag{1}$$

$$g(t) = \frac{\alpha}{\sqrt{\pi}} e^{-(\alpha t)^2}, \qquad \alpha = \pi B_t \sqrt{\frac{2}{\ln(2)}}, \tag{2}$$

where B_t is the single sided 3 dB bandwidth, and the output of the gaussian LPF is written as

$$\sum_{k=-\infty}^{\infty} a_k p_c(t-kT) \quad , \tag{3}$$

a_k are the binary information symbols taking values ± 1 with equal probability and $p_c(t)$ is the rectangular pulse response of the Gaussian filter.

$$\begin{aligned} p_c(t) &= (1/2)[erf\{\alpha t\} - erf\{\alpha(t-T)\}] \quad \text{K odd,} \\ &= \frac{1}{2}\left[erf\left\{\alpha\left(t+\frac{T}{2}\right)\right\} - erf\left\{\alpha\left(t-\frac{T}{2}\right)\right\}\right] \quad \text{K even,} \end{aligned} \tag{4}$$

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad , \tag{5}$$

Two different pulse shapes (one is the time shifted version of the other by T seconds) are defined for K odd and even cases for notational simplicity in analyzing psd. Since $p_c(t)$ is of infinite duration, and we can't compute psd in this case we truncate it to an interval of length KT seconds. This is also done in practise.

0-7803-5538-5/99/$10.00 (c) 1999 IEEE

Then the truncated pulse $p(t)$ is given by

$$p(t) = \frac{1}{2m}[erf\{\alpha t\} - erf\{\alpha(t-T)\}]$$
$$= \frac{1}{2m}\left[erf\left\{\alpha\left(t+\frac{T}{2}\right)\right\} - erf\left\{\alpha\left(t-\frac{T}{2}\right)\right\}\right] \qquad L_K \le t \le U_K \quad (6)$$

and

$$m = \int_{L_K}^{U_K} p(\mu)d\mu$$

$$L_K = \begin{cases} \frac{-(K-1)T}{2}, & \text{K odd} \\ -\frac{KT}{2}, & \text{K even} \end{cases} \qquad (7)$$
$$U_K = \begin{cases} \frac{(K+1)T}{2}, & \text{K odd} \\ \frac{K}{2}T, & \text{K even} \end{cases}$$

where L_K and U_K are the upper and lower limits of pulse duration and $p(t)$ is normalized such that the total area under the pulse is equal to T.

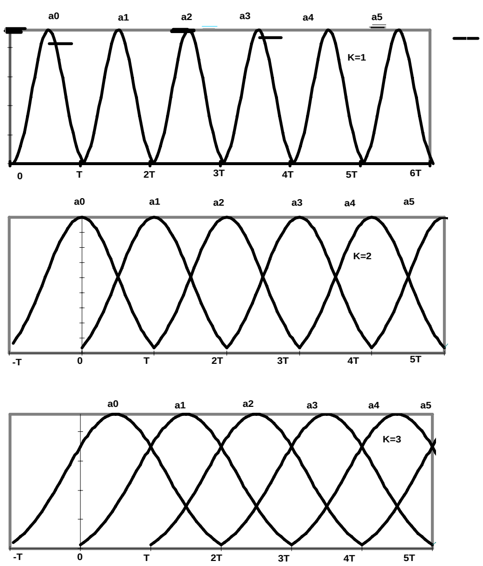

Figure 2. Modulation base band pulse train

For $B_bT = 0.3$ the value of m for different pulse truncation intervals is given in Table 1

Table 1. Area under the baseband pulse for $B_bT = 0.3$

K_0	**Area in%**
1	65.11
2	94.32
3	99.641
4	99.99
5	99.9999

The output of the modulator can be written as,

$$s(t) = \cos\left(2\pi f_c t + \frac{\pi\beta}{T}\int_{-\infty}^{t}\left\{\sum_{k=-\infty}^{\infty} b_k p(\mu - kT)\right\}d\mu\right), \qquad (8)$$

the modulation index β is set to 0.5

III. SPECTRAL DENSITY COMPUTATION

The GMSK signal represented in (8) is a binary digital frequency modulation (FM) with Gaussian pulse shaping and modulation index 0.5. We first consider deriving the psd of an Mary digital FM signal with arbitrary baseband pulse shaping, arbitrary modulation index and apply the resulting formulas to the special case of GMSK. Let

$$s(t) = \cos[2\pi f_c t + \phi(t) + \varphi], \qquad (9)$$

where $s(t)$ is a digital frequency modulation signal at a carrier frequency f_c, $\phi(t)$ is the modulation phase and φ is the random initial phase that is uniformly distributed between 0 and 2π. Let

$$\phi(t) = \int_{-\infty}^{t} f_d(t)d\mu \qquad (10)$$
$$= \int_{0}^{t} f_d(t)d\mu + \phi(0),$$

$\phi(0)$ is the phase of the signal at $t = 0$.

Define

$$v(t) = e^{j\phi(t)}, \qquad (11)$$

then

$$s(t) = Re\{e^{j2\pi f_c t}v(t)\}. \qquad (12)$$

$v(t)$ has periodic mean and periodic autocorrelation function with a period T. First, we calculate the spectral density of $v(t)$ and then we relate the spectrum of $s(t)$ to the spectrum of $v(t)$.

$$P_v(f) = \int_{-\infty}^{\infty} \Phi_v(\tau)e^{-j2\pi f\tau}d\tau, \qquad (13)$$

The power spectral density of $s(t)$ given in (12) can be written as,

$$P_s(f) = \frac{1}{4}[P_v(f-f_c) + P_v(f+f_c)]. \qquad (14)$$

IV. POWER SPECTRAL DENSITY OF $v(t)$

The power spectral density of $v(t)$ is derived in the appendix of [3]. It is shown that

$$P_v(f) = \frac{1}{T}[R(f)] \cdot [\tilde{P}_c(f)] \cdot [R(f)]^H, \qquad (15)$$

$$[R(f)] = \int_{-\infty}^{\infty}[r(t)]e^{-j2\pi ft}dt, \qquad (16)$$

the computation of $P_c(f)$ and $r(t)$ are explained elsewhere in this paper. Simplifying (15)

0-7803-5538-5/99/$10.00 (c) 1999 IEEE

$$P_v(f) = \frac{1}{T}[R(f)] \cdot [A + A^{\dagger}] \cdot [R(f)]^{\dagger} \ . \qquad (17)$$

A can be written in a generalized form [4] for any K

$$A = \frac{1}{2}[w_d]^{[K]} + \sum_{n=1}^{K-1} e^{-j2\pi fT}\{[q(U_K)] \cdot [w_d]^{[n]}\} \times [w_d]^{[K-n]} \times [w^{[n]}]^H + \frac{\{e^{-j2\pi fT}[w]^H \cdot [q(U_K)] \cdot w_d\}^{[K]}}{1 - e^{-j2\pi fT}[w]^H \cdot [q(U_K)]} \qquad (18)$$

$$[w_d]^{[K]} = [w_d] \times [w_d] \times \ \dots , \qquad (19)$$

where $[w_d]^{[K]}$ is the K th Kronecker product of the matrix $[w_d]$.

EXAMPLE: APPLICATION TO GMSK

Applying (17) and (18) to the case of GMSK, since we have equiprobable binary symbols,

$$w = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} , \qquad (20)$$

$$w_d = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} , \qquad (21)$$

for modulation index 0.5, the accumulated complex phase shift between adjacent symbols is given by

$$[q(U_K)] = \begin{bmatrix} e^{\frac{j\pi}{2}} & e^{\frac{-j\pi}{2}} \end{bmatrix} \ . \qquad (22)$$

For different pulse truncation lengths $[r(t)]$ can be defined as follows

GMSK-1

$$[r(t)] = [q(t)] \ , \qquad (23)$$

where

$$[q(t)] = \begin{bmatrix} e^{\frac{j\pi}{2T}\int_0^t p(\mu)d\mu} & e^{\frac{-j\pi}{2T}\int_0^t p(\mu)d\mu} \end{bmatrix} , \qquad (24)$$

$$p(t) = \begin{cases} \frac{1}{m}p_c(t), & 0 \le t \le T \\ 0 , & \text{else where} \end{cases} \qquad (25)$$

The notation GMSK-K represents GMSK with base band pulse truncated to K intervals.

GMSK-2

$$[r(t)] = [q(t)] \times [q(t-T)] \ , \qquad (26)$$

where

$$[q(t)] = \begin{bmatrix} e^{\frac{j\pi}{2T}\int_{-T}^t p(\mu)d\mu} & e^{\frac{-j\pi}{2T}\int_{-T}^t p(\mu)d\mu} \end{bmatrix} , \qquad (27)$$

$$p(t) = \begin{cases} \frac{1}{m}p_c(t) \ , & -T \le t \le T \\ 0 , & \text{else where} \end{cases} \qquad (28)$$

GMSK-3

$$[r(t)] = [q(t+T)] \times [q(t)] \times [q(t-T)] \ , \qquad (29)$$

where

$$[q(t)] = \begin{bmatrix} e^{\frac{j\pi}{2T}\int_{-T}^t p(\mu)d\mu} & e^{\frac{-j\pi}{2T}\int_{-T}^t p(\mu)d\mu} \end{bmatrix} , \qquad (30)$$

$$p(t) = \begin{cases} \frac{1}{m}p_c(t) \ , & -T \le t \le 2T \\ 0 , & \text{else where} \end{cases} \qquad (31)$$

GMSK-K

$$[r(t)] = \begin{cases} \prod_{\times\, i=-\frac{K+1}{2}}^{\frac{K+1}{2}} [q(t-iT)] \ , & \text{K odd} \\ \prod_{\times\, i=-\frac{K}{2}+1}^{\frac{K}{2}} [q(t-iT)] \ , & \text{K even} \\ 0 & \text{else where,} \end{cases} \qquad (32)$$

where

$$[q(t)] = \begin{bmatrix} e^{\frac{j\pi}{2T}\int_{L_K}^t p(\mu)d\mu} & e^{\frac{-j\pi}{2T}\int_{L_K}^t p(\mu)d\mu} \end{bmatrix} , \qquad (33)$$

and

$$p(t) = \frac{1}{m}p_c(t) \quad \begin{cases} \frac{-(K-1)T}{2} \le t \le \frac{(K+1)T}{2} \ , & \text{K odd} \\ -\frac{K}{2} \le t \le \frac{K}{2} + 1 \ , & \text{K even} \end{cases} \qquad (34)$$

$$m = \frac{1}{T}\int_{L_K}^{U_K} p(\mu)d\mu \ . \qquad (35)$$

V. RESULTS & DISCUSSION

GMSK psd is computed for various lowpass filter bandwidths and different amounts of pulse truncation. The spectrum can be obtained to any desired accuracy using matrix methods (accuracies of order -150 dB is obtained). The effect of pulse truncation length on the spectrum is examined. It is observed that incresed attenuation is obtained on the tails of the spectrum as the pulse truncation length is increased. For $B_bT = 0.16$ the main lobe of GMSK-6 is about the same as GMSK-8 where as the tails of GMSK-6 attenuate much faster than GMSK-2 and GMSK-8 (Figure 3). Note that the results for GMSK-8 (Figure 4) matches the Murata-Hirade's simulation results exactly. We also observed that the change in the psd is negligibly small for $K_0 > 8$. The system designer can choose the required pulse truncation length based on the out of band radiation requirements.

The spectral density results are plotted for different B_bT values for $K_0 = 8$ (Figure 4). The total band bandwidth (normalized with respect to T) required for 99.9% in band power is 1.132 and 2.76 for GMSK$(B_bT = 0.3)$ and MSK $(B_bT = \infty)$ respectively (Table 2). GSM system using GMSK modulation has 270 kHz bandwidth, 200 Kbps data rate and $B_bT = 0.3$. The out of band radiated power of such a system is less than -60 dB and that of a corresponding MSK system is -20 dB (Figure 5). This clearly illustrates the superiority of the GMSK method over MSK.

Table 2. Comparison of band occupancies for different pulse truncations at $B_bT = 0.3$

% of in band power	GMSK-2	GMSK-6	MSK
90%	0.646	0.602	0.78
99%	0.9642	0.91	1.2
99.9%	1.196	1.132	2.76
99.99%	1.892	1.599	6

Table 3. Comparison of band occupancies for different B_bT for GMSK-6

GMSK-6	$B_bT = 0.16$	$B_bT = 0.25$	$B_bT = 0.3$	$B_bT = 0.7$	MSK
90%	0.47	0.57	0.602	0.726	0.78
99%	0.72	0.86	0.91	1.08	1.2
99.9%	0.914	1.09	1.132	1.8	2.76

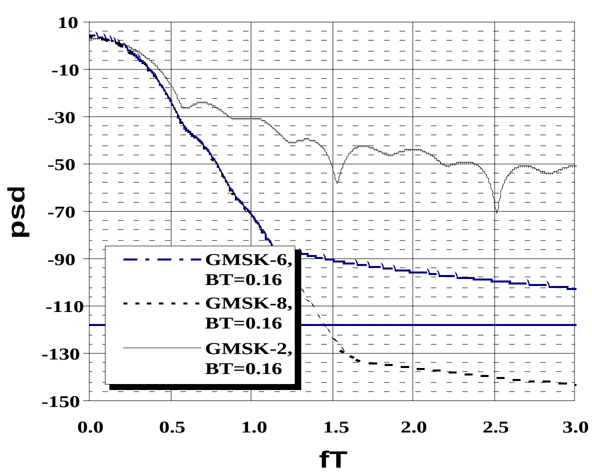

Figure 3. GMSK psd for different pulse truncation lengths

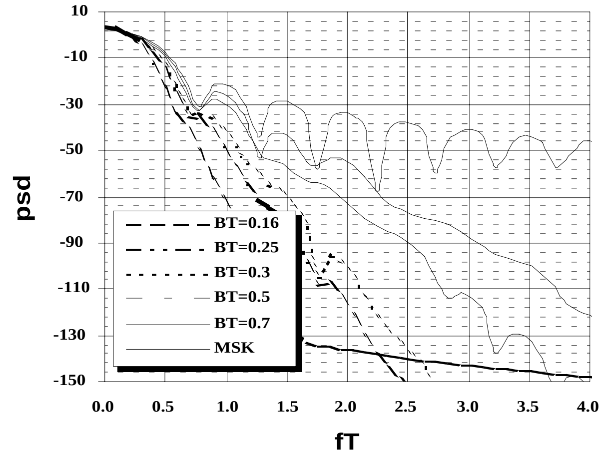

Figure 4. GMSK-8 psd for different B_bT

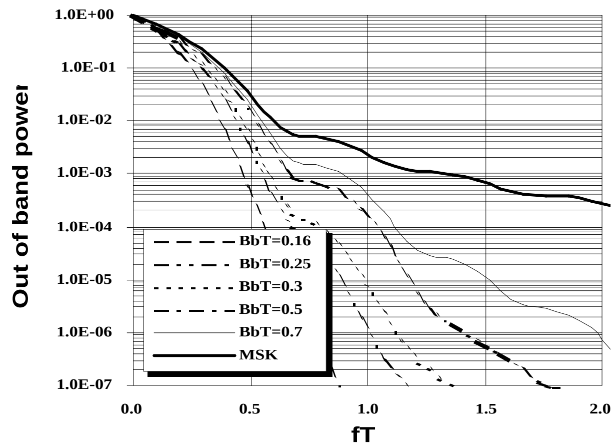

Figure 5. Total out of band power of GMSK-8

VI. CONCLUSIONS

We think that the matrix method of representing a GMSK signal is well suited not only for psd computation but also to evaluate the effect of ISI. We have clearly demonstrated that the matrix form of computation is not only superior to other methods of computation but also suited for high speed digital computers.

APPENDIX

VECTOR-MATRIX NOTATION

From (10) we write the frequency deviation signal as follows,

$$f_d(t) = \sum_{k=-\infty}^{\infty} h_{s_k}(t - kT) \,, \tag{36}$$

and $s_k = 1, 2, \ldots, M$.

$f_d(t)$ is a random baseband pulse train in which h_{s_k} can take one of the M possible waveforms $h_1, h_2 .., h_M$ of finite length K during the k^{th} time slot of duration T. The discrete random process s_k is assumed to be strictly stationary. The different signalling waveforms may overlap but are assumed to be independent i.e the random variable s_k is independent of s_l for $k \neq l$.

We rewrite (1) as

$$f_d(t) = \sum_{k=-\infty}^{\infty} [a_k^{(1)} h_1(t-kT) + \ldots + a_k^{(M)} h_M(t-kT)] \qquad (37)$$

for a given k (i.e., for a given time slot), one of the a_k' s is unity and the rest are zero

$$\begin{aligned} a_k^{(s_k)} &= 1, \\ a_k^{(i)} &= 0 \qquad i \neq s_k, \end{aligned} \qquad (38)$$

where s_k is the strictly stationary, discrete random process taking the integer values $1, 2, \ldots, M$.

Defining the following row vectors,

$$\begin{aligned} [a_k] &\equiv \left[a_k^{(1)}\ a_k^{(2)} \ldots\ a_k^{(M)}\right], \\ [h(t)] &\equiv \left[h_1(t)\ h_2(t) \ldots\ h_M(t)\right], \end{aligned} \qquad (39)$$

$$\begin{aligned} e_1 &\equiv \left[1\ 0\ 0 \ldots\ 0\right], \\ e_2 &\equiv \left[0\ 1\ 0 \ldots\ 0\right], \\ e_M &\equiv \left[0\ 0\ 0 \ldots\ 1\right], \end{aligned} \qquad (40)$$

then a_k can take one of the unit basis vectors $e_1, e_2, \ldots, e_M$ i.e,

$$a_k = e_{s_k}. \qquad (41)$$

we can write (37) in matrix notation as

$$f_d(t) = \sum_{k=-\infty}^{\infty} [a_k] \cdot [h(t-kT)]^H, \qquad (42)$$

where $\cdot$ denotes matrix multiplication throughout. The column vector $[h(t-kT)]^H$ is the transpose of the row vector $[h(t)]$. a_k are independent in this case but correlated $a_k's$ can be treated in a similar way.

STATISTICS OF THE VECTOR RANDOM PROCESS a_k

a_k is a vector valued, discrete random process, strictly stationary by the assumed stationarity of s_k. The mean and autocorrelation of the random process a_k are obtained in this section.

Define,

$$\begin{aligned} w_i &= Pr\{a_k = e_i\} = Pr\{s_k = i\}, \\ W_n(i, j) &= Pr\{a_k = e_i, a_{k+n} = e_j\} = Pr\{s_k = i, s_{k+n} = j\}, \end{aligned} \qquad (43)$$

w_i is the probability that the ith signalling waveform h_i is transmitted in any time slot, $W_n(i, j)$ is the joint probability that the signalling waveforms h_i and h_j are transmitted in two time slots separated by n signalling intervals. w_i and $W_n(i, j)$ are independent of k by the assumption of stationarity.

Let

$$w = \left[w_1\ w_2 \ldots\ w_M\right], \qquad (44)$$

be the probability row vector whose elements give the probabilities of the different signalling waveforms.

and

$$W_n = \begin{bmatrix} W_n(1,1) & W_n(1,2)\ldots & W_n(1,M) \\ W_n(2,1) & W_n(2,2)\ldots & W_n(2,M) \\ \ldots & \ldots & \ldots \\ W_n(M,1) & W_n(M,2)\ldots & W_n(M,M) \end{bmatrix}, \qquad (45)$$

be the matrix whose elements specify the joint probabilities of all pairs of signalling waveforms separated by n time slots. For $n = 0$, the joint probability matrix is diagonal, with diagonal elements the first order symbol probabilities

$$W_0 = \begin{bmatrix} w_1 & 0 & \ldots & 0 \\ 0 & w_2 & 0\ldots & 0 \\ \ldots & & w_3\ldots & 0 \\ 0 & \ldots & & w_M \end{bmatrix} = [w_d] \qquad (46)$$

then, the mean and autocorrelation of the strictly stationary, vector-valued, discrete random process a_k are,

$$\begin{aligned} E(a_k) &= [w], \\ \Phi_a(n) &= E(a_{k+n} \cdot a_k) = W_n, \end{aligned} \qquad (47)$$

In the case of independent a_k' s,

$$W_n(i, j) = w_i w_j \qquad n \geq 1. \qquad (48)$$

The spectral density of a_k is the discrete transform of (1), which is given by,

$$\tilde{P}_a(f) = \sum_{n=-\infty}^{\infty} W_n e^{-j2\pi fn}. \qquad (49)$$

$\tilde{P}_a(f)$ is a periodic function in f with unit period.

REPRESENTATION OF THE DIGITAL FM SIGNAL AS A BASEBAND PULSE TRAIN USING VECTOR- MATRIX NOTATION

The modulation signal is written as a baseband pulse train (in amplitude modulated form) using vector-matrix notation (See Figure 2). Writing $f_d(t)$ in matrix form

$$f_d(t) = \sum_{k=-\infty}^{\infty} [a_k] \cdot [h(t-kT)]^H, \qquad (50)$$

where a and h are M dimensional vectors. For signalling pulses $h_i(t)$ that are strictly time limited to an interval KT,

$$[h(t)] = [0] \qquad t < L_K, \qquad t > U_K. \qquad (51)$$

Then $v(t)$ can be written in amplitude modulation form as,

$$\begin{aligned} v(t) &= \sum_{k=-\infty}^{\infty} [c_k] \cdot [r(t-kT)]^H, \\ [c_k] &= S_k[b_k]. \end{aligned} \qquad (52)$$

The vectors b and r are M^K dimensional vectors expressed in terms of a and h., respectively. The different terms in (52) are strictly non-overlapping:

$$[r(t)] = [0] \qquad t < 0, \qquad t > T \,, \tag{53}$$

where b_k is a unit basis vector, i.e., one of its M^K components is unity and the rest are zero. The number of pulses contributing to $\phi(t)$ in each time slot equals K. Since each pulse can take on M different shapes, $\phi(t)$ can take on M^K different shapes in each time slot, the same is true for $v(t)$, thus demonstrating the representation of (52). The term S_k accounts for the memory (or accumulated phase due to integration) introduced in the signal due to the continuous phase nature of the modulation signal.

For any K, define

$$q(t) \equiv \left[e^{j\int_{L_K}^{t} h_1(\mu)d\mu} \; e^{j\int_{L_K}^{t} h_2(\mu)d\mu} \ldots e^{j\int_{L_K}^{t} h_M(\mu)d\mu} \right], \; L_k \le t \le U_k \qquad [0] \quad \text{else where} \tag{54}$$

$$[b_k] = \begin{cases} \displaystyle\prod_{\times\, i = -\frac{K+1}{2}}^{\frac{K+1}{2}} [a_{k+i}] & \text{K odd,} \\ \displaystyle\prod_{\times\, i = -\frac{K}{2}+1}^{\frac{K}{2}} [a_{k+i}] & \text{K even,} \end{cases} \tag{55}$$

$$[r(t)] = \begin{cases} \displaystyle\prod_{\times\, i = -\frac{K+1}{2}}^{\frac{K+1}{2}} [q(t-iT)] & \text{K odd,} \\ \displaystyle\prod_{\times\, i = -\frac{K}{2}+1}^{\frac{K}{2}} [q(t-iT)] & \text{K even,} \end{cases} \tag{56}$$

$$S_k = \begin{cases} \displaystyle\prod_{i=-\infty}^{k} \left[a_{i-\frac{(K+1)}{2}} \right] \cdot [q(U_K)]^H & \text{K odd,} \\ \displaystyle\prod_{i=-\infty}^{k} \left[a_{i-\frac{K}{2}} \right] \cdot [q(U_K)]^H & \text{K even.} \end{cases} \tag{57}$$

$\times$ denotes the right kronecker product and $\cdot$ denotes ordinary matrix multiplication. The right kronecker product of two matrices A and B with elements a_{ij}, b_{ij} is defined as,

$$A \times B \equiv \begin{bmatrix} a_{11}B & a_{12}B & \ldots \\ a_{12}B & a_{22}B & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}. \tag{58}$$

and

$$\prod_{\times\, i=L}^{M} A_i = A_L \times A_{L+1} \times \ldots \times A_{N-1} \times A_N \,. \tag{59}$$

DEFINITION OF $P_c(f)$

From (52)

$$v(t) = \sum_{k=-\infty}^{\infty} [c_k] \cdot [r(t-kT)]^H \,, \qquad [c_k] = S_k[b_k] \,. \tag{60}$$

$P_c(f)$ is the Fourier transform of the vector random process $[c_k]$.

$$\tilde{P}_c(f) = \sum_{n=-\infty}^{\infty} \tilde{\Phi}_c(n) e^{-j2\pi fn} \,, \tag{61}$$

where

$$\tilde{\Phi}_c(n) = \langle [c_{k+n}]^H \cdot [c_k{}^*] \rangle \,, \tag{62}$$

Simplification of $P_c(f)$ is given in [4].

REFERENCES

[1] K. Murota and K. Hirade, "GMSK Modulation for Digital Radio Telephony," IEEE Trans. Comm., COM-29, No. 7 (May 1978), pp. 1044-50.

[2] R. R. Anderson and J. Salz, "Spectra of Digital FM," B.S.T.J., 44, No.6 (July-August 1965), pp. 1165-1189.

[3] V. K. Prabhu and H. E. Rowe, Spectra of Digital PM by Matrix methods," B.S.T.J., 53, No.5 (May-June 1974), pp. 899-935.

[4] H. E. Rowe and V. K. Prabhu, "Power Spectrum of a Digital FM signal," B.S.T.J., 54, No. 6 (July-August 1975), pp. 1095-1125.

[5] V. K. Prabhu, "Spectral Occupancy of Digital Angle-Modulation Signals," B.S.T.J., 55, No. 4 (April 1976), pp. 429-453

[6] T. Aulin and C.-E. Sundberg, "Calculating Digital FM Spectral by Means of Autocorrelation," IEEE Trans. Comm., COM-30, No. 5 (May 1982), pp. 1199-1208.

[7] Kiran K. Kuchi, "Spectral Occupancy and Error Rate Considerations of GMSK Modulation for PCS Applications," Masters thesis, Dept. of Electrical Eng., The Univ. of Texas at Arlington, December 1997.