

# A FRACTAL MODULATION TECHNIQUE FOR DIGITAL COMMUNICATIONS SYSTEMS

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## ABSTRACT

*This paper addresses a method of fractal modulation for digital communications systems. The method has been developed for use in cases when a bit stream needs to be 'hidden' in transmission noise and can be used in addition to, or in place of a spread spectrum. This is achieved by embedding the information in data whose properties and characteristics resemble those of the background noise of a transmission system. The method is based on a random scaling fractal model.*

## INTRODUCTION

The application of fractal geometry for modelling naturally occurring signals and images is well known (e.g. [1], [2]). This is due to the fact that the 'statistics' and spectral characteristics of Random Scaling Fractals (RSFs) are consistent with many objects found in nature; a characteristic which is compounded in the term 'statistical self-affinity'. This term refers to random processes which have similar Probability Density Function (PDFs) at different scales. A RSF signal is one whose PDF remains the same irrespective of the scale over which the signal is sampled. As we zoom into a RSF signal, although the pattern of the signal (i.e. its time signature) changes, the PDF of the signal remains the same (a scaled down version of the original). Many noise types found in nature are statistically self-affine including a wide variety of transmission noise.

The purpose of Fractal Modulation is to try and make a bit stream 'look like' transmission noise. The problem is as follows: Given an arbitrary binary code, convert it into a RSF signal by modulating the fractal dimension of the RSF in such a way that the original binary code can be recovered in the presence of additive noise with minimal bit errors. The additional criteria that have been considered with regard to solving this problem are as follows: (i) The algorithm must produce a signal whose characteristics are compatible with a wide range of transmission noise; (ii) the algorithm must be invertible and robust in the presence of genuine transmission noise (with low Signal-to-Noise Ratios); (iii) the algorithm should ideally make use of conventional DSP technology, e.g. digital spectrum generation (FFT filters), real-time correlators (FIR filters).

## TRANSMISSION NOISE MODEL

There are two main criteria used to define the characteristics of a noise field: (i) The Probability Density Function (PDF) - the shape or envelope of the distribution of amplitudes of the field; (ii) The Power Spectral Density Function (PSDF) of the noise - the shape or envelope of the power spectrum. On the basis of these criteria, many noise fields have two fundamental properties: (i) The PSDF is determined by irrational power laws; (ii) The field is statistical self-affine. Two approaches are usually adopted in developing a stochastic model. The first is based on modelling the PDF of the system (or the Characteristic Function).

A pseudo random number generator is then designed to simulate the stochastic field. The second approach is based on modelling the PSDF. The stochastic field is then simulated by filtering white noise according to the PSDF model. In this paper, we consider the latter approach. Note that a 'good' stochastic model is one that accurately predicts both the PDF and the PSDF of the data. It is also one which takes into account the fact that a stochastic field may be non-stationary.

Consider the following non-stationary fractional differential equation

$$\left[ \frac{\partial^2}{\partial x^2} - \tau^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}} \right] u(x, t) = -f(x, t) \quad (1)$$

where  $0 \leq q(t) \leq 2 \forall t$ ,  $\tau$  is a positive constant and  $f$  is some stochastic source function which will be considered later. Here, non-stationarity is introduced through the use of a time varying fractional derivative which changes the physical meaning of the equation. This is different to conventional non-stationary modelling in which changes in the stochastic behaviour of  $u$  are introduced via the source function  $f$ . The parameter  $q$  is the 'Fourier Dimension' which is related to the Fractal (or Similarity) Dimension  $D$  for a signal by [2]  $D = q + \frac{1}{2}$ .  $1 < D < 2$ . Fractal modulation can now be defined in terms of  $q(t)$  (or  $D(t)$ ). It is a process whereby  $q(t)$  is assigned two states,  $q_0$  and  $q_1$  where  $q_0 \neq q_1$ . These states correspond to 0 and 1 in a bit stream respectively. The forward problem (fractal modulation) is then defined in terms of equation (1) (for fixed  $x$ ) as 'given  $q(t)$  compute  $u(t)$ ' and the inverse problem (fractal demodulation) is defined as 'given  $u(t)$  compute  $q(t)$ '. In this paper, we solve the forward problem using a Green function method, consider an asymptotic form to simplify the solution and then discretize the result. The inverse problem is then derived and solved in discrete form directly.

We shall consider a solution for constant  $q$  (corresponding to states  $q_0$  or  $q_1$ ) and separable  $f$ . In particular, let  $f(x, t) = \delta(x)n(t)$  where  $\delta$  is the delta function and  $n$  is (zero-mean) white Gaussian noise. For constant  $q$ , equation (1) can be

written in the form

$$\left( \frac{\partial^2}{\partial x^2} + \Omega_q^2 \right) U(x, \omega) = -\delta(x)N(\omega) \quad (2)$$

where  $U(x, \omega)$  and  $N(\omega)$  are the Fourier transforms of  $u(x, t)$  and  $n(t)$  respectively and  $\Omega_q^2 = -(i\omega\tau)^q$ . In obtaining this result, we have defined a fractional partial derivative as follows:

$$\frac{\partial^q}{\partial t^q} u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(x, \omega) (i\omega)^q \exp(i\omega t) d\omega$$

For the purpose of developing a transmission noise model, we shall now concentrate on the behaviour of  $u$  as a function of time. Using a Green function method [3] to solve equation (2) we can derive the asymptotic result (ignoring scaling)

$$U(\omega) \equiv \lim_{x \rightarrow 0} U(x, \omega) = \frac{N(\omega)}{(i\omega)^{q/2}}$$

Fourier-Laplace inversion then gives (using the convolution theorem [4] and ignoring scaling)

$$u(t) = \frac{1}{t^{1-q/2}} \odot n(t), \quad \text{Re}[q] > 0$$

where  $\odot$  denotes (causal) convolution. Note that  $u$  has the following fundamental property:  $\lambda^q \text{Pr}[u_\lambda(t)] = \text{Pr}[u(\lambda t)]$  where  $u_\lambda(t) = t^{(q/2)-1} \odot n(\lambda t)$ ,  $\lambda > 0$  and  $\text{Pr}[\cdot]$  denotes the PDF. This property describes the statistical self-affinity of  $u$ .

The digital algorithm for computing a discrete noise field  $u_k$  which is based on this model is as follows: (i) Compute a pseudo random array  $n_k, k = 0, 1, \dots, N-1$  whose spectrum is white; (ii) Compute the Discrete Fourier Transform (DFT) of  $n_k$  giving  $N_k$  using a Fast Fourier Transform (FFT); (iii) Filter  $N_k$  with  $1/(i\omega_k)^{q/2}$ ; (iv) Inverse DFT the result using an FFT to give  $u_k$ .

## INVERSE SOLUTION

Given the digital algorithm described above, the inverse problem can be defined thus: Given  $u_k$  compute  $q$ . A suitable approach to solving this

problem, which is at least consistent with the algorithm given in the last section is to estimate  $q$  from the power spectrum of  $u_k$  whose expected form (considering the positive half space only and excluding the DC term in which a singularity occurs) is  $\hat{P}_k = A\omega_k^{-q}$ ;  $k = 1, 2, \dots, (N/2)-1$  where  $A$  is a constant. Here, we assume that the FFT provides data in 'standard form' and that the DC or zero frequency component occurs at  $k = 0$ . Now consider the error function

$$e(A, q) = \|\ln P_k - \ln \hat{P}_k\|_2^2$$

where  $P_k$  is the power spectrum of  $u_k$ . Solving the equations (least squares method)  $\frac{\partial e}{\partial q} = 0$  and  $\frac{\partial e}{\partial A} = 0$  gives

$$q = \frac{N \sum_k (\ln P_k)(\ln \omega_k) - (\sum_k \ln \omega_k)(\sum_k \ln P_k)}{N \sum_k (\ln \omega_k)^2 - (\sum_k \ln \omega_k)^2}$$

and

$$A = \exp \left( \frac{\sum_k \ln P_k + q \sum_k \ln \omega_k}{N} \right)$$

The algorithm required to implement this inverse solution is as follows: (i) Compute the power spectrum  $P_k$  of the fractal noise  $u_k$  using an FFT; (ii) Extract the positive half space data (excluding the DC term); (iii) Compute  $q$  using the formula above. This algorithm provides a reconstruction for  $q$  (or alternatively  $D = q - \frac{1}{2}$ ) which is on average accurate to 2 decimal places for  $N \geq 64$  [2].

## FRACTAL MODULATION

The method of fractal modulation involves generating fractal signals in which two fractal dimensions are used to differentiate between 0 and 1 in a bit stream. Note that the fractal dimension is given by  $D = q + 0.5$  and for RSF signals has a value between 1 and 2. The technique is outlined as follows: (i) For a given bit stream, allocate  $D_0$  to bit=0 and  $D_1$  to bit=1; (ii) Compute a fractal signal of length  $N$  for each bit in the stream; (iii) Concatenate the results to produce a contiguous

stream of fractal noise  $u_k$ . One can increase the total number of samples by increasing the value of  $N$  (the number of samples per fractal) and/or increasing the number of fractals per bit. The information retrieval problem or fractal demodulation is solved by computing the fractal dimensions via the Power Spectrum Method discussed in the last section and using a conventional moving window to provide the fractal dimension signature  $D_j$ . The size of the window is taken to be equal to the number of samples per fractal. The binary sequence is then obtained from the following algorithm: If  $D_j \leq \Delta$  then bit=0, else bit=1 where  $\Delta = D_0 + \frac{1}{2}(D_1 - D_0)$ . The principal criteria for the optimization of this modulation technique is to minimize  $D_1 - D_0$  subject to accurate reconstructions for  $D_j$  in the presence of (real) transmission noise.

The software developed to investigate this modulation technique has been written with options on: (i) Samples per bit - number of samples used to compute a fractal signal; (ii) Fractals per bit - number of fractal signals used to represent one bit; (iii)  $D_0$  - fractal dimension for bit=0; (iv)  $D_1$  - fractal dimension for bit=1; (v) Addition of transmission noise before reconstruction.

An example of a fractal modulated signal is given in Figure 1 in which the binary code 0...1...0... has been considered in order to illustrate the basic principle. This figure shows the original binary code (top window) the fractal signal (middle window) and the fractal dimensions signature  $D_j$  (lower window - dotted line) using 1 fractal per bit, 64 samples per fractal for  $D_0 = 1.6$  and  $D_1 = 1.9$ . The reconstructed code is superimposed on the original code (top window - dotted line) and the original and estimated code is displayed on the right hand side. Note that there is a single error in the estimated code. By increasing the number of fractals per bit so that the bit stream is represented by an increased number of samples, greater accuracy can be achieved. In Figure 1, the difference in texture associated with 0 and 1 is clear because  $(D_0, D_1) = (1.1, 1.9)$ . By reducing  $(D_1 - D_0)$  this difference becomes negligible.

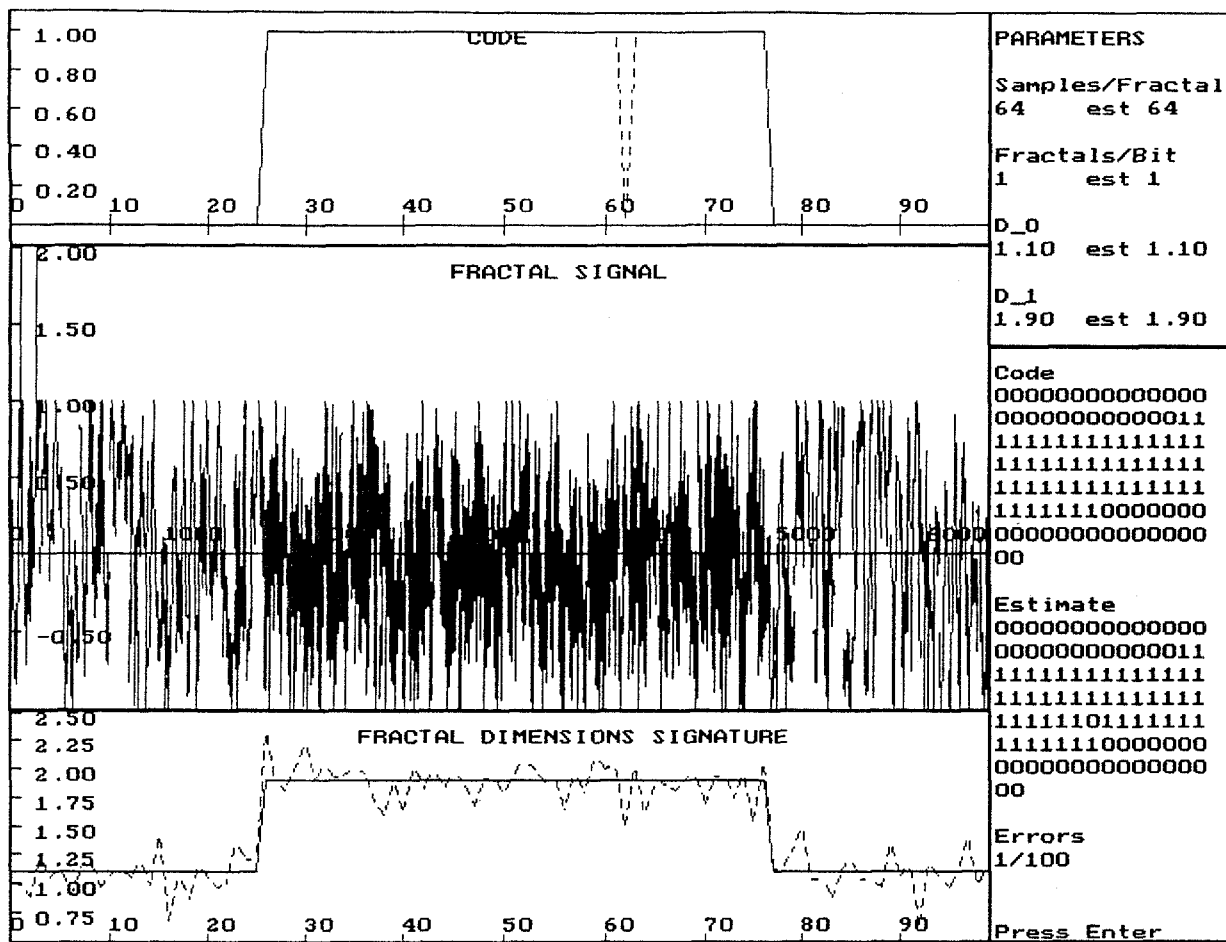


Figure 1. Fractal Modulation for 0...1...0...

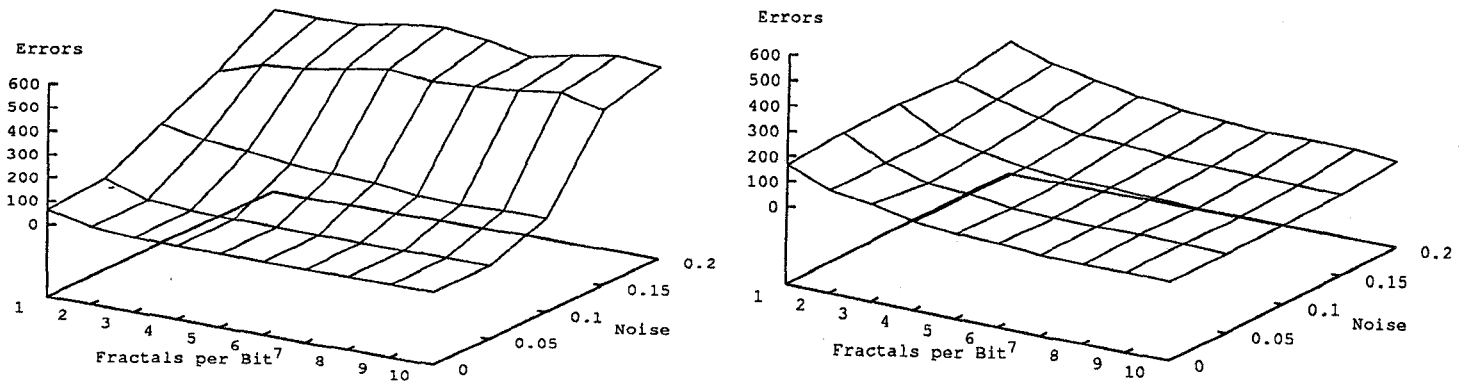


Figure 2. Example of bit errors as a function of 'Fractals per Bit' and 'Noise' for 64 samples per fractal with  $(D_0, D_1) = (1.1, 1.5)$  (left) and  $(D_0, D_1) = (1.5, 1.9)$  (right).

## EXAMPLE RESULTS

A detailed description of the results are beyond the scope of this paper but a short description of the approach and the overall trends are given. In order to obtain a quantitative picture of the accuracy of fractal demodulation subject to changes in the fractal generating parameters and additive noise, a bit stream of 1000 randomly chosen bits was used. The average number of errors (for 64 samples and then 128 samples) were compared with the number of 'Fractals per bit' and noise-to-signal ratio for different  $D_0$  and  $D_1$ . The noise  $n_k$  added to the RSF signal  $u_k$  was white Gaussian noise and the noise-to-signal ratio or 'Noise' defined in terms of the ratio  $\|n_k\|_\infty/\|u_k\|_\infty$ . Figure 2 provides some example surface plots showing the number of bit errors as a function of the number of fractals per bit and the noise, for fractals signals computed using 64 samples and different ( $D_0, D_1$ ). As expected, the results show that a combination of wide intervals between the two fractal dimensions with a large number of fractals per bit achieves greater accuracy. For example, the results for  $D_0 = 1.6$  and  $D_1 = 1.9$  achieves less than 3% bit error for 7 or more fractals per bit at 64 samples per fractal with 15% (i.e. 0.15) noise. In general, the bit error is less if  $\Delta$  is larger.

## CONCLUSIONS

Fractal modulation is a technique which attempts to embed a bit stream in fractal noise by modulating the fractal dimension. As expected, the error associated with recovering the bit stream is critically dependent on the SNR. The reconstruction algorithm provides relatively low error rates with a relatively high level of noise, provided the difference in fractal dimension is not too small and that many fractals per bit are used. In any application, the parameter settings would have to be optimized with respect to a given transmission environment. The technique could work with lower SNRs if coupled with a suitable inference engine.

The success of the technique (with regard to its covert intent) depends on the appropriateness of the transmission noise model used to embed a bit

stream. In this paper we have used a model compounded in equation (1) which leads to a PSDF of the type  $\omega^{-q}$ . This power law is consistent with statistically self-affine noise but is not ideally suited to all noise. Another possible PSDF is [5]  $|U(\omega)|^2 = A\omega^{2g}(\omega_0^2 + \omega^2)^{-g}$  where  $q$  and  $g$  are positive (floating point) numbers and  $A$  is a constant. This PSDF represents a more general and possibly, a more versatile model. It is consistent with a wider range of noise than the one considered here but it also poses a significantly more difficult inverse problem [6]. Another possible extension to the fractal modulation technique considered in this paper is to choose a larger number of states  $q_n, n = 1, 2, \dots$  representing a (renormalised) run length code for example.

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