The General Power Variance Method for Direction Finding

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Abstract

In this paper, we propose and analyze a method of of estimating the angle of arrival of the dominant component of a signal. Our approach is based on solving a class of optimization problems involving the moments of the directional power variance. This method contains as a special case the power variance method. Simulations are used to illustrate our approach. Generalizations of the method are also discussed.

1 Introduction

In this paper, we use a class of functionals related to the moments of the directional power variance to study the angle of arrival of the dominant component of a signal from an antenna array.

We model the received signal as a sum of transmitted signals and background noise by

$$r = s_1 u_1 + \cdots + s_m u_m + w,$$

where the signals $s_1, \ldots, s_m$ are unknown real or complex valued random variables, the directions $u_1, \ldots, u_m$ are fixed unit vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$, and the background noise $w \in \mathbb{R}^n$ or $\mathbb{C}^n$ is a random variable. We denote the variances of $s_1, \ldots, s_m$ by $\sigma_1^2, \ldots, \sigma_m^2$ and assume that $\sigma_1^2 \geq \cdots \geq \sigma_m^2$. The problem is to estimate u_1. We assume that the number of distinguishable signals is less than the number of antenna elements, that is, $m \leq n$. If the density of s_1 is symmetric, which is generally the case, then $s_1 u_1$ and $-s_1 u_1$ have the same distribution, and so u_1 and $-u_1$ cannot be distinguished from the received signal alone. Therefore we will only attempt to estimate u_1 up to an 180° ambiguity.

2 The General Power Variance Method

The General Power Variance method uses a class of functionals, representing the performance measure, that contains the power variance measure proposed in [2] as a special case. The performance measures are indexed by p, $1 \leq p \leq \infty$. We will show that for the case with two dominant signals, the direction that minimizes the measure for $p = \infty$ is *exactly* orthogonal to the dominant signal direction. We will also give results that indicate that as $p \to \infty$, the p-measure approximates the ∞-measure. The theoretical analysis of the performance of this method is complicated and the analytical results incomplete for $1 \leq p < \infty$. Simulations show that the method works well for $p \geq 4$. This approach should be analyzed further.

Let

$$\|X\|_p = [\mathrm{E}(|X|^p)]^{1/p} = \left(\int_\Omega |X(\omega)|^p d\mu(\omega)\right)^{1/p}$$

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for $1 \leq p < \infty$, and for $p = \infty$,

$$\|X\|_\infty = \sup_{\omega \in \Omega} |X(\omega)|.$$

The following is well-known (see, for example, [3]):

Theorem 2.1 *For any random variable X,*

(i) if $1 \leq p \leq q \leq \infty$, then $\|X\|_p \leq \|X\|_q$, and

(ii) $\lim_{p\to\infty} \|X\|_p = \|X\|_\infty$.

The idea behind the original power variance method is that if we minimize the power variance of the received signal over all directions, the directions that give the minimum should be almost orthogonal to the dominant signal. More precisely, for each $z \in \mathbb{C}^n$, $\|z\| = 1$, form the random variable

$$X_z(\omega) = |z^* r(\omega)|^2 - \mathrm{E}(|z^* r|^2),$$

where r is the received signal. It is suggested in [1] and [2] that the solution of

$$\begin{aligned} &\min_{\|z\|=1} \|X_z\|_2 \\ = &\min_{\|z\|=1} \left(\mathrm{E}\left[\left(|z^*r|^2 - \mathrm{E}(|z^*r|^2) \right)^2 \right] \right)^{1/2} \end{aligned}$$

should point in a direction that is almost orthogonal to the direction of the dominant signal. The claim is motivated by physical considerations ([1]) and is supported by simulation results. Motivated by the positive results reported there, we are led to consider

$$\begin{aligned} &\min_{\|z\|=1} \|X_z\|_p \\ = &\min_{\|z\|=1} \left(\mathrm{E}\left[\left| |z^*r|^2 - \mathrm{E}(|z^*r|^2) \right|^p \right] \right)^{1/p} \end{aligned}$$

for $1 \leq p < \infty$ and

$$\begin{aligned} &\min_{\|z\|=1} \|X_z\|_\infty \\ = &\min_{\|z\|=1} \max_{\omega} \left| |z^*r(\omega)|^2 - \mathrm{E}(|z^*r|^2) \right| \end{aligned}$$

for $p = \infty$. For $1 \leq p \leq \infty$, let M_p be the set of solutions of the minimization problem. That is, for $1 \leq p \leq \infty$,

$$\begin{aligned} M_p = \quad & \{w \in \mathbb{C}^n : \|w\| = 1\} \\ \bigcap \quad & \{\|X_w\|_p = \min_{\|z\|=1} \|X_z\|_p\}. \end{aligned}$$

Note that by the compactness of $\{\|z\| = 1\}$, M_p is nonempty and that the minimization problem for p has a unique solution precisely when M_p is a singleton set. In general, the minimization problems have unique solutions. Clearly, if z is a solution then $-z$ is also a solution and so there cannot be a unique solution in the usual sense. We use "unique" in a sense consistent with our assumption that we do not distinguish the vectors z and $-z$ when discussing directions. Since for any random variable X,

$$\|X\|_p \to \|X\|_\infty,$$

we expect that if $z_p \in M_p$ and $z_p \to z$, then $z \in M_\infty$. Indeed, it is shown in [4] that this is true for the two dominant signals case. Therefore, the estimates that we obtain for large p's approximate solutions for $p = \infty$. In the next section, we show that for the two dominant signal case, the solution for the minimization problem when $p = \infty$ is orthogonal to the direction of a signal under mild assumptions.

3 The Two Signal Case for $p = \infty$

We now analyze the the performance of the power variance method for the $p = \infty$ case. Under mild assumptions, we show that for the special case of two signals, the solution of the minimization problem is orthogonal to the direction of a signal. In this section, we assume that

1. $r = s_1 u_1 + s_2 u_2$, where u_1, u_2 are independent unit vectors;
2. s_1, s_2 have continuous density functions with support $[-k_1, k_1]$ and $[-k_2, k_2]$, respectively.

Under these assumptions, we have

Theorem 3.1 *If $k_j^2 \geq 2\sigma_j^2$, for $j = 1, 2$, then*

$$\begin{aligned} &M_\infty = \\ &\left\{ u_j^\perp : k_j^2 - \sigma_j^2 \leq \min(k_1^2 - \sigma_1^2, k_2^2 - \sigma_2^2) \right\}. \end{aligned}$$

Proof: See [4].

Remark 3.1 *For example, it follows from the theorem that if $k_2^2 - \sigma_2^2 < k_1^2 - \sigma_1^2$, then the unique solution to $\inf_{\|z\|=1} \|X_z\|_\infty$ is $u_2^\perp$.*

What random variables satisfy the hypothesis of the Theorem? It is not hard to show that if s is a real random variable with continuous unimodal density function with support $[-k, k]$, then $k^2 \geq 3\sigma_s^2$. In particular, this is true for the uniform density function. It can also be shown that if we consider the sinusoid $s = k\sin(2\pi ft)$ as a random variable, then $k^2 = 2\sigma^2$.

From the Theorem, we have the following corollary,

Corollary 3.2 *For $i = 1, 2$, let s_1, s_2 be uniformly distributed on $[-k_i, k_i]$. Let $r = s_1u_1 + s_2u_2$, where u_1, u_2 are independent unit vectors. If $k_1 > k_2$, then the unique solution of $\inf_{\|z\|=1} \|X_z\|_\infty$ is $u_1^\perp$.*

The theorem also applies to sinusoidal signals as follows. We can consider without loss of generality a sinusoid to have the form

$$s(t) = k\sin(2\pi ft).$$

The signal $s(t)$ can be consider as a random variable on the interval $[-k, k]$. Clearly $s(t)$ has zero mean and its variance, or power, can be computed:

$$\begin{aligned}
\sigma_s^2 &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} k^2\sin^2(2\pi ft)\,dt \\
&= k^2 \lim_{T\to\infty} \frac{1}{2T}\frac{1}{2\pi f}\left(\frac{t}{2} - \frac{\sin t\cos t}{2}\right)_{-2\pi fT}^{2\pi fT} \\
&= \frac{k^2}{2}.
\end{aligned}$$

The hypothesis of the theorem is satisfied.

Before proceeding further, we give an example to show that some restriction must be imposed on the density function for the results of this section to hold. Let s_i take the values $-k_i$ and k_i with equal probability and let $r = s_1u_1 + s_2u_2$. The covariance matrix is

$$C = k_1^2u_1u_1^* + k_2^2u_2u_2^*.$$

It can be shown that

$$\|X_z\|_\infty = z^*Cz$$

and so

$$\inf_{\|z\|=1} \|X_z\|_\infty = \inf_{\|z\|=1} z^*Cz = \lambda_{\text{min}},$$

with the solution in the direction of the eigenvector corresponding to λ_{min}. This eigenvector is orthogonal to the dominant eigenvector, which is in general not in the direction of the dominant signal. We see that for this class of signals, the power variance method for $p = \infty$ is basically equivalent to the Signal Separator. By using a continuous approximation to the above discrete density, we see that there are continuous densities for which the above results do no apply.

The next theorem shows that we can use the solutions of the minimization problems for $p < \infty$ to approximate the true direction. Simulations show that $p = 4$ and $p = 8$ are sufficient in many cases.

Theorem 3.3 *Assume the hypothesis of Theorem 3.1. Let z_p be a solution of $\inf_{\|z\|=1} \|X_z\|_p$ for each $1 \leq p < \infty$. Then all cluster points of the sequence $\{z_p\}$ are solutions of $\inf_{\|z\|=1} \|X_z\|_\infty$. In particular, if $\inf_{\|z\|=1} \|X_z\|_\infty$ has a unique solution z_∞, then $z_p \to z_\infty$.*

Proof: See [4].

4 Simulations

In this section, we illustrate our approach with computer simulations results. We use a model of the received signal of the form

$$r = s_1u_1 + s_2u_2$$

where s_1, s_2 are symmetric uniformly distributed and u_1, u_2 are unit vectors with u_1 in the direction of 0° and u_2 at 45°. For all the simulations, s_1 is uniform on the interval $[-k, k]$, where k is a parameter, and s_2 is uniform on the interval $[-2, 2]$. We have

$$\sigma_1^2 = \frac{k^2}{3} \text{ and } \sigma_2^2 = \frac{4}{3}.$$

and

$$C = \frac{k^2}{3}u_1u_1^* + \frac{4}{3}u_2u_2^* = \frac{1}{3}\begin{pmatrix} k^2+2 & 2 \\ 2 & 2 \end{pmatrix}.$$

For the power variance method, we use $k = 2.5, 3, 4, 6$ and $p = 2, 4, 8, \infty$. For each trial run, we generate 100 sample points of the received signal using the given distributions of s_1 and s_2 and numerically find the solution z_p for each p. For the simulation, 100 runs are used. The 100 direction estimates obtained

$p\backslash\beta_\sigma(dB)$	1.9	3.5	6.0	9.5
2	8.8	5.5	2.8	1.2
4	4.3	2.7	1.3	0.6
8	2.2	1.4	0.7	0.2
∞	1.2	0.8	0.3	0.2
S.S.	16.3	12.0	7.0	3.2

Table 1: Dominant Direction Estimation Errors-Uniform Signals

$p\backslash\beta_\sigma(dB)$	1.9	3.5	6.0	9.5
2	13.2	8.5	4.7	1.9
4	9.0	5.2	2.9	1.1
8	5.6	3.2	1.7	0.7
∞	2.9	1.6	0.9	0.3
S.S.	16.3	12.0	7.0	3.2

Table 2: Dominant Direction Estimation Errors-Uniform Signals

for each p from the trials are averaged to give an estimate of an orthogonal direction of u_1. The estimates for the direction of u_1 are obtained by subtracting 90° from the orthogonal estimates. We also estimate the direction of u_1 using the dominant eigenvector for the same values of k. The estimates we obtained using the dominant eigenvectors are computed exactly and are the values we would obtain if we had averaged the estimates of a large number of trials each using an estimate of the covariance matrix C computed from data samples. The results obtained are summarized in Table 1. The entries in the first row are the values of $\beta_\sigma = \sigma_1^2/\sigma_2^2$ in dB's for $k = 2.5, 3, 4, 6$ and the entries in the first column are the values of p. The estimates obtained using the Signal Separator are in the last row. Other than the first row and first column, the entries are in degrees. Since the true direction is 0°, the estimates are also the estimation errors. It is clear from Table 1 that, for uniformly distributed signals, the power variance method performs much better than the Signal Separator.

5 Generalizations

In the Power Variance method, the p-norm of

$$X_z(\omega) = |z^*r(\omega)|^2 - \mathrm{E}(|z^*r|^2)$$

is considered. We can also use the functions

$$Y_z(\omega) = |z^*r(\omega)|^2.$$

The only difference between X_z and Y_z is that the $\mathrm{E}(|z^*r|^2)$ term has been dropped. Under less stringent conditions, results similar to those in the previous section can be derived (see [4]). For the two dominant signal case, we have the following theorems.

Theorem 5.1 *If $k_1 > k_2$, then the solution of $\inf_{||z||=1} ||Y_z||_\infty$ is $u_1^\perp$. If $k_1 = k_2$, then $u_1^\perp, u_2^\perp$ are the solutions.*

Theorem 5.2 *If $k_1 > k_2$, then the solutions of $\inf_{||z||=1} ||Y_z||_p$ converge to $u_1^\perp$ as $p \to \infty$.*

Simulations show that this approach gives estimates comparable to those given by the Power Variance method. The results, obtained by using the same parameters as in the Power Variance case, are summarized in Table 2. The true direction is 0°. The simulations show what we expect from the theory. The estimates for the $p = \infty$ case are close to the true direction for all the k_i's used and the accuracy improves with increasing p. In all cases, the estimates are better than the signal separator estimates. A comparison of the results of this method with the power variance method shows that the power variance method is more accurate for simulations with the same parameters. However, this method gives the exact theoretical orthogonal to the dominant signal direction when $p = \infty$ for all bounded signals.

Another approach is to use the solutions of

$$\min_{||z||=1} \left[\mathrm{E}\left(||r||^2 - |z^*r|^2\right)^p\right]^{1/p}.$$

This is equivalent to the Signal Separator when $p = 1$. If the individual signals are bounded and independent, the locus of the received signal is a parallelepiped. It can be shown that for $p = \infty$

$$\min_{||z||=1} \max_{\omega} \left(||r(\omega)||^2 - |z^*r(\omega)|^2\right)$$

is in the direction of the median along the longest side of the parallelepiped. In general, the median along

$p\backslash\beta_\sigma(dB)$	1.9	3.5	6.0	9.5
2	12.7	8.6	4.6	1.8
4	8.2	5.4	2.7	1.0
8	4.8	3.3	1.6	0.5
∞	2.2	1.6	0.7	0.1
S.S.	16.1	12.0	7.1	3.0

Table 3: Dominant Direction Estimation Errors-Uniform Signals

the longest side of the parallelepiped corresponds to the direction of the dominant signal. As in the case of the power variance method, the solution for $p = \infty$ is exact and the solutions for the finite p's tend to the correct direction as $p \to \infty$.

We illustrate this method with simulations. The results are summarized in Table 3.

6 Conclusions

In this paper, we gave three similar methods for estimating the angle of arrival of the dominant signal. The General Power Variance method is motivated by the Power Variance method of Bond and Schmidt and contains it as a special case. The second method is obtained by changing the terms of the functionals used in the General Power Variance method, and the third method is motivated by geometric considerations. All three methods use the solutions of optimization problems as the estimates. It is shown that the solutions of the optimization problem give the exact directions in special cases. The methods are illustrated by simulations.

References

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