UPPER AND LOWER BOUNDS ON THE CUT-OFF RATE FOR CODED TRANSMISSIONS OVER TIME-CORRELATED RAYLEIGH-FADED MOBILE-RADIO LINKS

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ABSTRACT

This paper presents new upper and lower bounds on the symmetric cut-off rate for coded Quadrature-Amplitude-Modulated (QAM) signalling over mobile radio links affected by time-correlated Rayleigh distributed flat faded phenomena. The proposed bounds assume Maximum Likelihood (ML) soft-decoding with perfect Channel-State-Information (CSI) at the receiving side and hold for *any format* of QAM constellations. These bounds are *quickly* computable and constitute an efficient means for estimating the cut-off rate when the assumption of perfect interleaving falls short. Analytical and numerical evidence of the tightness of the presented bounds is also provided for some typical mobile radio links.

I. INTRODUCTION

Data-transmission over mobile radio channels is generally affected by fading phenomena which can present high time-correlations, as in the case of terrestrial links between low-speed units moving over urban areas [2,3]. Due to the limits imposed on the overall allowable decoding-delay, such time-correlation cannot be in general fully removed by interleaving/deinterleaving devices; thus, as a consequence of the memory present in the coding-channel, the resulting cut-off rate of the system *depends* on the length of the transmitted codewords and its evaluation presents a computational complexity which grows *exponentially* with the adopted block-length (see [2] and [3] for the cases of discrete and continuous coding-alphabet, respectively).

The goal of this contribution is twofold. Firstly, in Sect. II we present a general formula for the (exact) computation of the cut-off rate of flat-faded Rayleigh-links which generalises the expression reported in [2,eq.(14)] and holds for *any format* of QAM constellations, like those which are asymmetric and with non-constant envelope largely employed, for example, in spectrally-efficient coordinate-interleaved systems [4]. Secondly, in Sect. III we present tight easily computable lower and upper bounds which constitute an attractive tool for a quick estimation of the cut-off rate of the considered data-systems. The presented bounds are new; alternative bounds do not seem available in literature for the kind of channels here examined (see, for example, [3,5] and references therein for an extensive overview of known bounds on the cut-off rate for channels with memory).

II. EVALUATION OF THE SYMMETRIC CUT-OFF RATE FOR BI-DIMENSIONAL CONSTELLATIONS

Let us assume that an M-ary information stream feeds an encoder whose output $\mathbf{c}\equiv[c(1)\ldots c(N)]\in A^N$ is a codeword composed of N bi-dimensional symbols taken from an assigned q-ary constellation A. Thus the corresponding discrete-time sequence $\{r(i)\in \mathbf{C}^1, 1\le i\le N\}$ received at the output of a noisy link affected by flat-fading is given by [2]

$$r(i) = g(i)c(i) + \eta(i), \quad 1 \le i \le N, \tag{1}$$

where $\{\eta(i)\}$ is a complex zero-mean Gaussian noise sequence with variance N_0. Furthermore, in eq.(1) $\{g(i)\equiv g_c(i)+jg_s(i)\in \mathbf{C}^1\}$ is a zero-mean stationary Gaussian fading sequence whose uncorrelated components share a common autocorrelation function (a.c.f.) $\{R_g(t), 0\le|t|\le N-1\}$; thus, the resulting N-variate Gaussian fading-vector $\mathbf{G}\equiv[g(1)\ldots g(N)]^T$ exhibits a real covariance matrix $\mathrm{Cov}_\mathbf{G}\equiv[\mathrm{cov}_\mathbf{G}(k,m), 1\le k,m\le N]$ whose scalar entries are defined as: $\mathrm{cov}_\mathbf{G}(k,m)\equiv 2R_g(t=|k-m|)$.

Now, under the assumption of perfect CSI, the ML soft-decoder present at the receiver' site decides for the codeword which minimizes the usual Euclidean-distance from the available received vector $\mathbf{r}\equiv[r(1)\ldots r(N)]$; hence, moving from an application of the union-Battacharyya bound and then using random-coding argumentations, it can be proved that the symmetric⁽¹⁾ cut-off rate $R_0^*(N)$ for the depicted communications-system is given by the general formula [1,eqs.(5.6.1),(5.9.35)], [2,eq.(7)]

$$R_0^*(N) = 2\,lg\,q - \frac{1}{N}\,lg\left\{\int\limits_{\mathbf{r}\in\mathbf{C}^N}\right. \left.\int\limits_{\mathbf{G}\in\mathbf{C}^N}\left[\sum_{\mathbf{c}\in A^N}\sqrt{p(\mathbf{r}\,|\,\mathbf{c},\mathbf{G})}\right]^2 p(\mathbf{G})\,d\mathbf{G}\,d\mathbf{r}\right\}, \tag{2}$$

where $p(\mathbf{r}|\mathbf{c},\mathbf{G})$ denotes the Gaussian probability density function (p.d.f.) of the received vector **r** conditional on **c** and **G**, whereas $p(\mathbf{G})$ is the Gaussian p.d.f. of the fading-vector **G** (hereafter, natural logarithms are used). The N-fold integrations present in (2) can be carried out by resorting to standard formulas for the computation of multivariate Gaussian integrals [2,eq.(B.5) and following text] and the final result appears thus [6]:

$$R_0^*(N) = 2\,lg\,q - \frac{1}{N}\,lg\left\{q^N + \right. \left. + \sum_{\mathbf{c}\in A^N}\ \sum_{\substack{\mathbf{c}'\in A^N\\ \mathbf{c}'\neq\mathbf{c}}}\Phi\left(\mathbf{c},\mathbf{c}';N_0,\mathrm{Cov}_{\mathbf{G}}\right)\right\}, \tag{3}$$

with

$$\Phi\left(\mathbf{c},\mathbf{c}';N_0,\mathrm{Cov}_{\mathbf{G}}\right) \equiv \equiv \left\{det\left[\mathrm{I} + \frac{1}{4N_0}\Delta^2(\mathbf{c},\mathbf{c}')\mathrm{Cov}_{\mathbf{G}}\right]\right\}^{-1}, \tag{4}$$

where

$$\Delta^2(\mathbf{c},\mathbf{c}') \equiv \mathrm{diag}\left\{\left\|c(i) - c'(i)\right\|^2\ ,1 \le i \le N\right\}, \tag{5}$$

is the $N\times N$ diagonal matrix which gathers the squared Euclidean distances between the N symbols of the codewords **c** and **c**'. The formula in (3) holds for *any form* of QAM constellation and reduces to the expression (14) of [2] when symmetric constant-envelope constellations are considered. Obviously, as it happens for the formula (14) of [2], even the evaluation of (3) presents a computational complexity which grows *exponentially* with the block-length N. In fact, for general QAM constellations the evaluation of (3) requires the calculation of $0.5(q^{2N}-q^N)$ N^{th}-order determinants in (4) so that the cut-off rate computation results cumbersome even for moderate values of N and q. Such a drawback can be effectively overcome through the exploitation of the bounds presented in the next Section.

III. THE PROPOSED UPPER AND LOWER BOUNDS FOR THE SYMMETRIC CUT-OFF RATE $\mathbf{R_0^*(N)}$

Let us indicate with d^2 and D^2 the minimum and maximum squared Euclidean distances between a pair of constellation points. By developing the inner summation present in (3) and then suitably upper-bounding the determinants in (4) we get the upper-bound for the cut-off rate in (3) reported in eq. (6) at the

$$\overline{R_0^*}(N) \equiv lg\,q - \frac{1}{N}\,lg\left\{1 + N(q-1)\left(1+\frac{\mathrm{D}^2}{2N_0}R_g(0)\right)^{-1} + (q-1)^2\,\frac{N(N-1)}{2}\left[1+\frac{\mathrm{D}^2}{N_0}R_g(0) + \right.\right. \left.\left. + \left(\frac{\mathrm{D}^2}{2N_0}\right)^2\left(R_g^2(0) - \left|R_g(min;N-1)\right|^2\right)\right]^{-1} + \left[q^N - 1 - N(q-1)\left(1+\frac{(q-1)(N-1)}{2}\right)\right]\frac{1}{det\left[\mathrm{M}(N_0;N)\right]}\right\} \tag{6}$$

$$\underline{R_0^*}(N) \equiv lg\,q - \frac{1}{N}\,lg\left\{1 + N(q-1)\left(1+\frac{\mathrm{d}^2}{2N_0}R_g(0)\right)^{-1} + (q-1)^2\,\frac{N(N-1)}{2}\left[1+\frac{\mathrm{d}^2}{N_0}R_g(0) + \right.\right. \left. + \left(\frac{\mathrm{d}^2}{2N_0}\right)^2\left(R_g^2(0) - R_g^2(1)\right)\right]^{-1} + \left[q^N - 1 - N(q-1)\left(1+\frac{(q-1)(N-1)}{2}\right)\right]\left[\left(1+\frac{\mathrm{d}^2}{2N_0}R_g(0)\right)^3 + \right. \left.\left. +2\left(\frac{\mathrm{d}^2}{2N_0}\right)^3 R_g^2(1)\,R_g(2) - \left(1+\frac{\mathrm{d}^2}{2N_0}R_g(0)\right)\left(\frac{\mathrm{d}^2}{2N_0}\right)^2\left(2R_g^2(1)+R_g^2(2)\right)\right]^{-1}\right\} \tag{7}$$

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(1) According to a current taxonomy [5], the term “symmetric” means that independently-selected and equi-distributed codeword-symbols are assumed in the derivation of the formula in (2).

bottom of the page, with the positions $\left|R_g\left(min; N-1\right)\right| \equiv \quad \equiv \min_{1\le t\le N-1}\left\{ \left|R_g\left(t\right)\right| \right\}$ and $\mathsf{M}\left(N_0 ; N\right) \equiv \quad \equiv \mathrm{I} + \left({}^{1}\!\big/_{4N_0} \right) \mathrm{D}^2 \mathrm{Cov}_{\mathbf{G}}$.

Furthermore, by lower-bounding the same determinants in (4) we also obtain the lower-bound for the cut-off rate in (3) below reported in eq. (7) [6, Appendix].

The above bounds represent the main result of this contribution and their properties are pointed out in the following remarks.

Remark 1. The computational complexity of the lower-bound in (7) is virtually independent of the values of *N* and *q* whereas the evaluation of the upper-bound in (6) essentially requires the calculation of the determinant of the *N*×*N* matrix $\mathsf{M}(N_0;\mathrm{N})$. However, since the latter is a symmetric positive-definite Toëplitz-type matrix, the computation of its determinant can be *quickly* accomplished through the standard Levinson-Durbin algorithm [7]. So, especially for large values of *N* and *q*, the computational burden for evaluating the presented bounds is nearly negligible with respect to that needed for the computation of (3).

Remark 2. Since the following limiting expressions hold:

$$\lim_{N_0\to 0} \overline{R_0^*}(N) = \lim_{N_0\to 0} \underline{R_0^*}(N) \equiv \lim_{N_0\to 0} R_0^*(N) = lg\, q \tag{8}$$

$$\lim_{N_0\to \infty} \overline{R_0^*}(N) = \lim_{N_0\to \infty} \underline{R_0^*}(N) \equiv \lim_{N_0\to \infty} R_0^*(N) = 0 \tag{9}$$

we can conclude that the proposed bounds are certainly tight for high and low levels of the average received SNR per channel-symbol $\overline{\gamma}_c$. Furthermore, the numerical examples of Figs. 1,2,3 directly support the good tightness exhibited by the bounds even for moderate values of $\overline{\gamma}_c$. In fact, from the plots reported in Figs.1,2,3 we can conclude that the difference between the presented bounds and the actual cut-off rate values typically falls below 7%-10%.

Remark 3. It can be analytically proved that the values assumed by the bounds in (6), (7) increase when the time-correlation of the fading-phenomena falls and a direct comparison of the plots reported in Figs. 2, 3 confirms this property. So, in this respect the behaviour of the presented bounds *closely mimics* that exhibited by the actual cut-off rate in (3).

Remark 4. From the foregoing it could be expected that a good “estimate” of the cut-off rate in (3) is given by the arithmetical average

$$\tilde{R}_0^*\left(N\right) \equiv \frac{1}{2}\left[\overline{R_0^*}\left(N\right) + \underline{R_0^*}\left(N\right)\right] , \tag{10}$$

of the proposed bounds. Although an analytical proof of this assertion is in general hard to obtain, the numerical examples of Figs.1,2,3 indeed support this claim; in fact, an examination of the reported plots shows that the estimate in (10) differs from the actual value in (3) by 2%-4% so that the corresponding curves of Figs.1,2,3 appear nearly indistinguishable. Therefore, on the basis of the above remarks we can conclude that the computation of (10) constitutes a *quick means* to accomplish a tight evaluation of the (exact) cut-off rate value given by the formula in (3).

References

[1] R.G. Gallager, *Information theory and reliable Communication*, J. Wiley and Sons, 1968.

[2] K.L.-Boullé, J.C. Belfiore, “The Cut-Off Rate of Time-Correlated Fading Channels”, *IEEE Trans. on Inform. Theory*, vol. 39, no. 2, pp. 612-617, March 1993.

[3] G. Kaplan, S. Shamai, “Achievable Performance over Correlated Rician Channel”, *IEEE Trans. on Comm.*, vol. 42, no. 11, pp. 2967-2978, November 1994.

[4] B.D. Jelicic, S. Roy, “Cut-Off Rates for Coordinate Interleaved QAM over Rayleigh Fading Channels”, *IEEE Trans. on Comm.*, vol. 44, no. 10, pp. 1231-1233, October 1996.

[5] S. Shamai, A. Dembo, “Bounds on the Symmetric Binary Cut-Off Rates for Dispersive Gaussian Channels”, *IEEE Trans. on Comm.*, vol. 42, no. 1, pp. 39-53, January 1994.

[6] E. Baccarelli, “Bounds on the Cut-Off Rates of Faded Channels with Memory”, *INFOCOM Dpt. Tech. Rep. 003-001-97*, Univ. “La Sapienza”, Rome 1997.

[7] J.G. Proakis, *Digital Communications*, 2nd Ed., McGraw-Hill, pp.137-139.

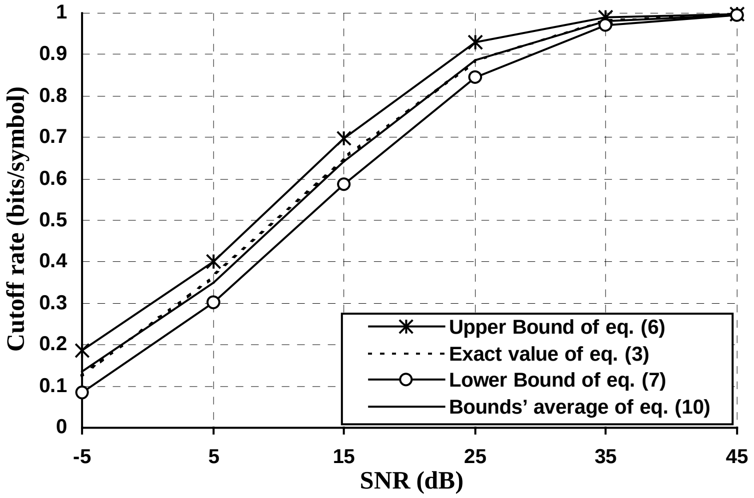

Fig. 1: Behaviour of eqs. (3), (6), (7), (10) versus the average received SNR per channel-symbol $\bar{\gamma}_c$ for the mobile radio link of eq. (1) affected by Rayleigh-faded phenomena with time-correlation described by the usual zero-order Bessel-function [2]: $R_g(t) \equiv J_0(2\pi B_D T t)$. Codeword-length N of 32 and BPSK constellation are considered for a value of the product Doppler-spread×signalling-period $B_D T$ of 10^{-4} (case of a *slow-faded* mobile radio link).

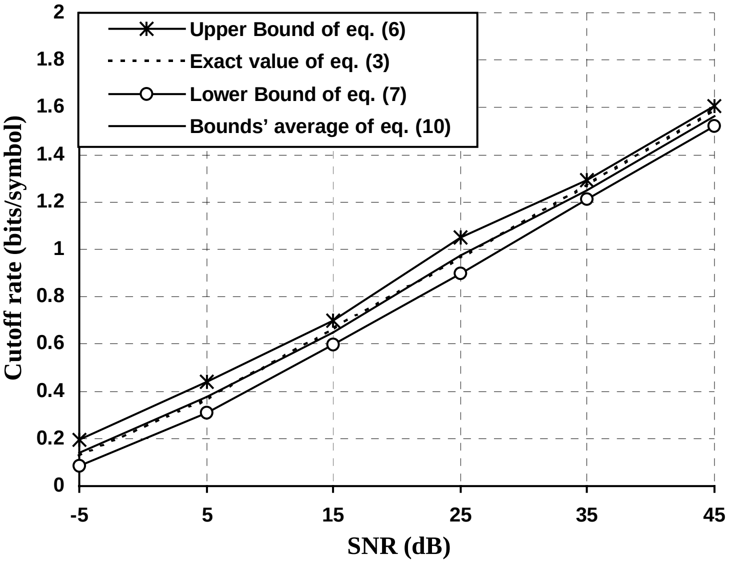

Fig 2: The same as in Fig. 1 for a 4PSK constellation (N=32 and $B_D T$=10^{-4}).

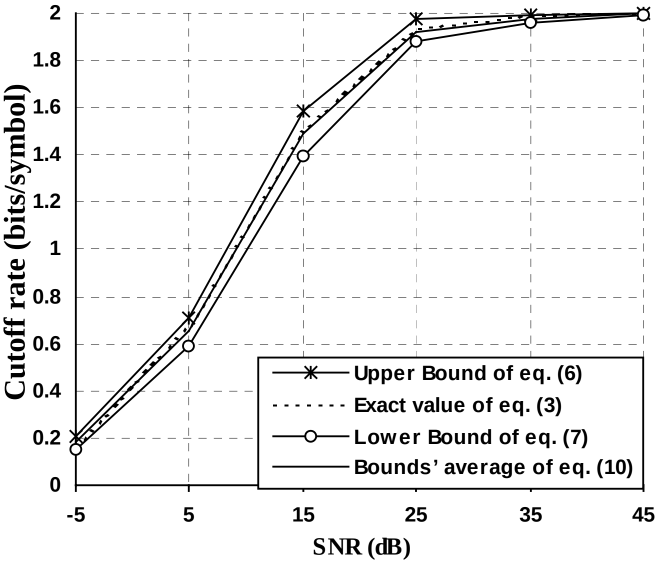

Fig. 3: The same as in Fig. 2 for $B_D T$=$8\cdot10^{-2}$ (N=32 and 4PSK constellation).