TCMs Matched to ISI Channels:

New Results for Combined Symbol-by-Symbol Equalization and Decoding

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Abstract: The design of good TCM schemes matched to ISI channels requires that analytical performance bounds are also available. In this paper, starting from an application of the union-Bhattacharyya bound, upper-bounds for the performance of the theoretically optimum combined Symbol-by-Symbol (SbS) Abend-Fritchman like equalizer and decoder are presented for the first time in literature. The combined detector acts on the distorted version of the TCM signal received at the output of a channel impaired by (known) ISI and AWGN. On the basis of these bounds a novel simple criterion for the design of TCMs optimally matched to any assigned ISI-corrupted transmission channel is then derived and its application to some actual test-channels is illustrated.

I. Motivation of the Work

Although the high bandwidth efficiency of TCMs makes their utilization very attractive for data-transmissions over band-limited ISI-impaired channels, to date little has been done about analytical evaluation of TCMs performance on ISI channels, as pointed out in [1,Sect.11.1] and more recently in [2], neither general criteria for the good design of TCMs matched to these channels seem available.

The set of bounds recently proposed in [3] well represent the state of the art about this topic. However, the interesting results of [3] have been derived under some simplifying assumptions [3,Sect.III-IV],[4] which could make the resulting bounds loose when not perfectly interleaved systems acting over severely ISI degraded environments are considered. Furthermore, as also explicitly pointed out by the same authors in [3,Sect.IX], *no criterion* for designing TCMs is developed.

The results of [3] (as those of [1,Chap.11] and [4,5]) also assume the presence of Maximum Likelihood Sequence (MLS) receivers with additive decoding metrics implemented via the standard Viterbi Algorithm (VA). However, an attractive alternative to MLS-VA detectors is today constituted by the so called SbS receivers, originally developed by Abend and Fritchman [6,Sect.6.6], and recently reconsidered with renewed interest [7,8,9]. The utilization of SbS equalizers and decoders seems indeed very attractive for data-transmission over severely ISI-impaired links; in fact, several simulation results for TCM-encoded data systems have proved that the performances obtained by means of SbS detectors can be over *two orders* of magnitude better than those achievable with conventional MLS-VA receivers [7]. However, despite its practical as theoretical relevance the question concerning analytical results about the key-parameters dominating the performance of TCM-encoded SbS-decoded ISI-impaired data systems appears today unresolved; more in detail, an examination of the current open literature shows that the following topics result still unexplored.

1) Barring the recent contribution in [10], no expressions for the analytical evaluation of SbS equalizers/decoders acting on TCM-encoded and ISI-impaired signals are available.

2) No criterion is known for the optimal design of TCMs matched to ISI channels when SbS detectors are employed (as previously remarked, this question appears still substantially unresolved even for the conventional MLS-VA receivers).

By extending and generalizing some previous results valid for uncoded systems [10], answers to the above listed questions are presented in this contribution. SbS receivers which optimally combine the two tasks of equalization and decoding and act on the ISI-distorted noisy versions of the transmitted TCM signals are considered *without resorting* to the simplifying assumption of perfect interleaving; as it is well known, due to their combined structure the considered receivers are theoretically optimum because they really minimize the decoding Symbol Error Probability (SEP) [6,Sect.6.6],[7,8].

II. The Model of the Communication System

The discrete-time complex baseband equivalent data transmission system we consider is reported in Fig.1, where the discrete-time ISI channel accounts for the combined effects of transmitting filter, noisy time-dispersive analog waveform channel, whitened-matched receiving filter and baud-rate sampler [6,Sec.6.3].

The m-bit information vector $a(i) \in A_A \equiv \left\{\alpha_0, \ldots, \alpha_{L_A-1}\right\}$, $L_A \equiv 2^m$, is assumed to be a L_A-ary random variable (r.v.) with outcomes taking values on the L_A-ary alphabet $A_A \in GF(2^m)$ (therein after, GF(q) indicates a Galois field with q elements). The resulting i.i.d. information stream $\{a(i),\ i \geq 1\}$ is mapped into the coded modulated sequence $\left\{s(i) \in A_S \equiv \left\{s_0, \ldots, s_{L_E-1}\right\} \subset C^1,\ i \geq 1\right\}$, $L_E \equiv 2^{m+l}$, by a TCM of rate $R_E \equiv m/(m+l)$ and constraint-length L_E. As it is well known [1,Sect.3.4], the TCM can be described as a Moore type finite state machine (FSM) which operates according to the relations (see Fig.1)

$$\sigma_E(i+1) = \varphi_E(\sigma_E(i); a(i)), \quad (1.a)$$

$$s(i) = \psi_E(\sigma_E(i)), \quad (1.b)$$

where $\varphi_E(\cdot;\cdot)$ is the so called “state-transition function” of the encoder, $\psi_E(\cdot)$ is the “output transformation” and [6,Sect.5.3], [11,Sect.4.1]:

$$\sigma_E(i) \equiv \left[a(i)\ a(i-1) \ldots a(i-L_E+1)\right]^T \in GF(2^{mL_E}), \quad (2)$$

(1) This work was partially supported by European Union under the ACTS AC104 MOSTRAIN project.

is the mL_E-bit vector of the state of the encoder at the i-th epoch ($i\geq1$). Thus, the ISI-corrupted noisy random sequence observed at the input of the combined SbS channel-equalizer and TCM-decoder can be modeled as ($i\geq1$)

$$y(i)=\sum_{k=0}^{L_C-1} g(k)s(i-k)+v(i)\equiv \mathbf{G}^{\mathbf{T}}\mathbf{s}(i)+v(i)\equiv z(i)+v(i), \quad (3)$$

where $\mathbf{G}\equiv\left[g(0)\ ...\ g(L_C-1)\right]^T\in C^{L_C}$ is the assigned L_C-long impulse response vector of the ISI channel, $\mathbf{s}(i)\equiv\left[s(i)\ ...\ s(i-L_C+1)\right]^T\in\left(A_S\right)^{L_C}$ is the state-vector and $\left\{v(i)\right\}$ is a complex zero mean Gaussian noise sequence whose uncorrelated components exhibit a common variance equal to ($N_o/2$). Thus, the cascade of the TCM encoder and the ISI channel can be modeled as a Moore-type FSM [11,Sect.4.10],[5,Sect.2] of constraint length $L_T\equiv L_C+L_E$-1. The state of this machine (for $i\geq1$)

$$\mathbf{x}(\mathrm{i})=\left[a(i)\ ...\ a(i-L_T+1)\right]^T\in A_X\equiv(A_A)^{L_T}, \quad (4)$$

is a discrete $N_X\equiv 2^{mL_T}$-ary r.v. with equally likely outcomes taking values on the set $A_X\equiv\left\{\xi_0,\ ...,\ \xi_{N_X-1}\right\}$ with elements defined as in the following:

$$\xi_j\equiv\left[\alpha_0^{(j)}\ ...\ \alpha_{L_T-1}^{(j)}\right]^T\in GF(2^{mL_T}),\ 0\leq j\leq N_X-1,\ \left(\alpha_i^{(j)}\in A_A\right); \quad (5)$$

therefore, the resulting super-state sequence $\{\boldsymbol{x}(i)\in A_X,\ i\geq1\}$ constitutes a stationary N_X-ary Markov-chain [5, Sect. 2].

The input to the above considered super-machine is the information stream $\{a(i)\}$ while the corresponding output is the sequence $\{z(i)\equiv\mathbf{G}^{\mathbf{T}}\mathbf{s}(i)\in C,\ i\geq1\}$ of eq.(3) observed at the output of the noise-free portion of the ISI channel (see Fig.1). The latter sequence can be directly related to the super-state sequence on the basis of the relation: $z(i)=\mathbf{G}^{\mathbf{T}}\mu_o(\mathbf{x}(i))$, where the L_C-variate complex function $\mu_o(\cdot)$: $A_X\rightarrow C^{L_C}$ is defined as

$$\mu_o(\mathbf{x}(i))=\left[\psi_E(\vartheta(0)\otimes\mathbf{x}(i)),\ ...,\ \psi_E(\vartheta(L_C-1)\otimes\mathbf{x}(i))\right]^T \quad (6)$$

(in eq.(6), the symbol $\otimes$ indicates that the underlying summations and products must be carried out bit-wise according to the usual algebraic rules over GF(2)). In eq.(6) the binary matrices $\vartheta(k)$, $0\leq k\leq L_C-1$, have elements on GF(2) and must satisfy the relations (see eqs.(2),(4)): $\sigma_E(i-k)=\vartheta(k)\otimes\mathbf{x}(i)$; so, they are block-matrices of L_E row-blocks and L_T column-blocks structured as in the following:

$$\vartheta(k)\equiv\begin{array}{c}\left|\begin{array}{ccc:cccc:ccc}
\mathbf{0}_{\mathrm{mxm}} & \mathbf{0}_{\mathrm{mxm}} & \cdots & \mathbf{I}_{\mathrm{mxm}} & \mathbf{0}_{\mathrm{mxm}} & \cdots & \mathbf{0}_{\mathrm{mxm}} & \mathbf{0}_{\mathrm{mxm}} & \cdots & \mathbf{0}_{\mathrm{mxm}}\\
\vdots & \cdots & \cdots & \mathbf{0}_{\mathrm{mxm}} & \mathbf{I}_{\mathrm{mxm}} & \cdots & \mathbf{0}_{\mathrm{mxm}} & \vdots & \cdots & \vdots\\
\vdots & \cdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots\\
\vdots & \cdots & \cdots & \vdots & \vdots & & \vdots & \vdots & \cdots & \vdots\\
\mathbf{0}_{\mathrm{mxm}} & \mathbf{0}_{\mathrm{mxm}} & \cdots & \mathbf{0}_{\mathrm{mxm}} & \mathbf{0}_{\mathrm{mxm}} & \cdots & \mathbf{I}_{\mathrm{mxm}} & \mathbf{0}_{\mathrm{mxm}} & \cdots & \mathbf{0}_{\mathrm{mxm}}
\end{array}\right|\\
\begin{array}{cc}\uparrow\ k\text{ - th column} & \uparrow\ (k+L_E-1)\text{ - th column}\end{array}\end{array}$$

Therefore, from the foregoing we have that the observation of eq.(3) can be equivalently rewritten as

$$y(i)=\mathbf{G}^T\mu_o(\mathbf{x}(i))+v(i),\ i\geq1, \quad (7)$$

where $\mu_o(\cdot)$ acts as the output-function of the super-machine of Fig.1 and is related to the super-state $\mathbf{x}(i)$ through the assigned output function $\psi_E(\cdot)$ of the employed TCM (see eq.(6)).

Now, as it well known a combined SbS equalizer and decoder with non-negative integer decision-delay D generates the estimated m-bit vector sequence $\{\hat{a}(i-D)\in A_A,\ i\geq1\}$on a step-by-step basis according to usual MAP decision rule [6,Sect.6.6]. Obviously, the performance of any MAP detector *does not depend* on the employed implementation of the decision algorithm, so that the specification of this last is indeed immaterial for our purposes (see [7,8,9,10] for recent contributions on this topic). It is also well known that an application of the "Total Probability Theorem" allows us to relate the A Posteriori Probabilities (APPs) of the transmitted information stream $\{a(i)\}$ to those pertaining to the super-state sequence $\{\mathbf{x}(i)\}$ of eq.(4) as

$$P\left(a(i-r)=\alpha_k\,|y_1^{i+\Delta}\right)=$$
$$=\sum_{\xi_j\in A(r,k)} Pr\left(\mathbf{x}(i)=\xi_j\,|y_1^{i+\Delta}\right),\ 0\leq r\leq L_T-1,\ 0\leq k\leq L_A-1, \quad (8)$$

where $\Delta\equiv D-r$, $0\leq r\leq L_T$ -1, is the assigned delay for decoding the super-state sequence $\{\mathbf{x}(i)\}$, $y_1^{i+\Delta}$ is the available realization of the observation sequence from step 1 to step ($i+\Delta$), and, finally, $A(r,k)$ is the sub-set of A_X constituted by the determination $\xi_j\in A_X$ whose r-th vector element is equal to $\alpha_k\in A_A$.

III. SEP Upper-Bounds for Combined SbS Equalization and Decoding

The results of the previous section constitute the basic tool requested to develop the desired bound. At this regard, we begin to note that an application of the usual union-Bhattacharyya bound [11,Sect.2.3] allows to dominate the SEP of the combined SbS equalizer and decoder as

$$P_e(i-r|i+\Delta)\equiv P\left(a(i-r)\neq\hat{a}(i-r|i+\Delta)\right)\leq\frac{1}{L_A}\sum_{m=0}^{L_A-1}\sum_{\substack{j=0\\ j\neq m}}^{L_A-1}$$
$$\int_{y_1^{i+\Delta}\in C^{i+\Delta}}\sqrt{P\left(y_1^{i+\Delta}|a(i-r)=\alpha_m\right)P\left(y_1^{i+\Delta}|a(i-r)=\alpha_j\right)} \quad (9)$$

Now, a suitable exploitation of relation (8) allows us to compute the ($i+\Delta$)-fold integrals present in (9) by resorting to standard results about the integration of multivariate Gaussian distribution [12,eq.(4.118)]. So, it can be proved that the bound assumes the following final form

$$P_e(i-r|i+\Delta)\leq 2^{-2m(i+\Delta-1+L_T)}\sum_{j=0}^{L_A-1}\sum_{\substack{j\neq m\\ m=0}}^{L_A-1}$$
$$\left\{\sum_{t\in A(r,m)}\ \sum_{n\in A(r,j)}\ \sum_{x_1^{i+\Delta}\in S_X(i+\Delta;i;t)}\ \sum_{\bar{x}_1^{i+\Delta}\in S_X(i+\Delta;i;n)} Z^{d_o^2(x_1^{i+\Delta};\bar{x}_1^{i+\Delta})}\right\} \quad (10)$$

where $Z\equiv\exp(-1/(4N_o))$ is the so called Bhattacharyya parameter and $d_o^2(\cdot;\cdot)$ are Euclidean distance parameters defined as:

$$d_o^2(x_1^{i+\Delta};\bar{x}_1^{i+\Delta})\equiv\sum_{l=1}^{i+\Delta}\left\|\mathbf{G}^{\mathbf{T}}\left[\mu_o(\mathbf{x}(l))-\mu_o(\bar{\mathbf{x}}(l))\right]\right\|^2.$$

Furthermore, about eq.(10) the following positions also hold:

- $x_1^{i+\Delta}, \bar{x}_1^{i+\Delta}$ indicate two determinations allowed to the $(i+\Delta)$-long random sequence of super-states $\{\mathbf{x}(i)\}$ from step 1 to step $(i+\Delta)$;
- $\mathbf{x}(l), \bar{\mathbf{x}}(l)$ are the l-th components of the $(i+\Delta)$-long determinations $x_1^{i+\Delta}$ and $\bar{x}_1^{i+\Delta}$, respectively;
- $S_X(i+\Delta;i;t)$ is the sub-set of allowed determinations $x_1^{i+\Delta}$ with i-th component equal to $\xi_t \in A_X$ (an analogous definition holds for $S_X(i+\Delta;i;n)$ with the t-index replaced by the n-index).

The simulation results in Sect.V test the tightness of the presented bound. About the meaning of the parameter $d_o^2(\cdot;\cdot)$, we note that this last can be rewritten as $d_o^2(x_1^{i+\Delta};\bar{x}_1^{i+\Delta})=\sum_{l=1}^{i+\Delta}\left\|z(\mathbf{x}(l))-\bar{z}(\mathbf{x}(l))\right\|^2$; so, it can be viewed as the squared Euclidean distance between the two $(i+\Delta)$-long sequences $z_1^{i+\Delta}(x_1^{i+\Delta})$ and $z_1^{i+\Delta}(\bar{x}_1^{i+\Delta})$ received at the output of the super-machine of Fig.1 in correspondence of the two determinations $x_1^{i+\Delta}$ and $\bar{x}_1^{i+\Delta}$ of the $(i+\Delta)$-long random sequence of the super-states, respectively.

IV. A Simplified Upper-Bound and a TCM Design Criterion

The bound developed in eq.(10) generally depends on $L_A(L_A-1)2^{2m[i+\Delta+L_T-2]}$ distinct distance parameters so that it appears still too complex to be effectively employed for TCM design purposes. In this section we present a simpler (but looser) bound depending on a *single* suitable defined minimum distance parameter which summarizes the ultimate performance of combined SbS equalizers and decoders for $N_o\to 0$. More in particular, starting from the inequality of eq.(10) it can be proved that the following more simple bound holds for $N_o\to 0$:

$$P_e(i-r|i+\Delta)\le\left(1-\frac{1}{L_A}\right)Z^{d_{min}^2(r)},\ 0\le r\le L_T-1,\ \Delta\ge 0 \quad (11)$$

where $d_{min}^2(r)$ indicates the minimum element of the set $H(r)$ defined as $0\le m<j\le L_A-1$

$$H(r)\equiv\left\{\left\|\mathbf{G}^{\mathbf{T}}\left[\mu_o(\xi_t)-\mu_o(\xi_n)\right]\right\|^2,\ t\in A(r,m),\ n\in A(r,j)\right\} \quad (12)$$

The simpler bound of eq.(11) appears quite tight for medium to high signal to noise ratio (SNR), as the simulation results of Sect.V confirm. Furthermore, on the basis of eq.(12) it can be viewed that the parameter $d_{min}^2(r)$ represents the minimum squared Euclidean distance between two signal samples $z(\mathbf{x}(i)=\xi_t)$ and $z(\mathbf{x}(i)=\xi_n)$ received at the output of the super-machine of Fig.1 when the realizations ξ_t and ξ_n of the super-state $\mathbf{x}(i)$ disagree *at least* in the r-th component (see eq.(5)).

Now from the definition of eq.(12) it follows that the elements of the set $H(r)$ (and thus corresponding minimum value $d_{min}^2(r)$) jointly depend on the transmitting property of the ISI channel through the impulse vector $\mathbf{G}$ and also on the transformation $\psi_E(\cdot)$ performed by the employed TCM through the mapping function $\mu_o(\cdot)$ (see eq.(6)). Therefore, on the basis of the bound (11) we can claim that a simple and effective criterion for the design of good TCMs matched to ISI channels *consists to select the output transformation $\psi_E(\cdot)$ of the encoder so to maximize the minimum distance parameter $d_{min}^2(r)$.*

Because the size of the set $H(r)$ is finite and equal, at most, to $0.5(1-1/L_A)N_X^2$, the minimum search procedure embedded into the stated optimality criterion appears, in general, *not* very cumbersome, so that quite simple computer-aided procedures can be easily implemented for design purposes.

V. Simulation Results

To test the tightness of the proposed bounds of eqs.(10)-(11), we have considered the ISI channels listed in Tab.I and the TCM constituted by a convolutional encoder of R_E=1/2 with octal generators (7,5) [6,Chap.V] followed by a signal-mapping into a QPSK constellation performed according to the usual Gray rule. The measured BERs reported in Figs.2-3 have been obtained through Montecarlo simulations; at the receiver side, combined SbS equalization and decoding with decision delay equal to the overall memory-length (L_T-1)=5 of the system has been implemented according to Sect.III of [7]. An examination of the results of Fig.2 show that the bound of eq.(10) generally differs from the corresponding BER less of about 0.9, 0.6 and 0.4 dB at measured BERs of about 10^{-3}, 10^{-5} and 10^{-6}, respectively. The simplified bound of eq.(11) is indeed looser than that of eq.(10); however, for increasing SNR, the bounds differ for less than about 0.4-0.3 dB for BER values below 10^{-3}.

To illustrate the design criterion presented in Sect. IV, we have considered the two-taps channel B of Tab. I and we have searched good encoders over the simple class of TCMs with R_E=1/2, L_E=2 with QPSK constellations. So, it is easy to show that in this case the set $H(2)$ of eq.(12) is constituted by fourteen distinct elements and that the corresponding minimum element $d_{min}^2(2)$ is proportional to the squared Euclidean distance: $\left\|\psi_E(\sigma_0)-\psi_E(\sigma_3)\right\|^2$, where $\sigma_o\equiv[0\ 0]^T$, $\sigma_1\equiv[1\ 0]^T$, $\sigma_2\equiv[0\ 1]^T$, $\sigma_3\equiv[1\ 1]^T$. Thus, in this case the stated design criterion requires that the encoder output function $\psi_E(\cdot)$ is selected so to maximize the above minimum distance and a possible function satisfying the requirement can be defined as in the following:

$$\begin{aligned}&\psi_E(\sigma_0)\equiv+j;\ \psi_E(\sigma_1)\equiv+1;\\&\psi_E(\sigma_2)\equiv-1;\ \psi_E(\sigma_3)\equiv-j.\end{aligned} \quad (13)$$

From Fig.3 it can be viewed that the optimized TCM (although *very* simple) gives rise to coding gains of about 0.6 dB and 0.9 dB at BERs of 10^{-5} and 10^{-6}, respectively. This supports the conclusion that even simple TCMs suitably matched to ISI channels can give rise to good coding gains when combined SbS equalization and decoding is carried out at the receiver side *without* requiring the presence of interleaving/deinterleaving structures.

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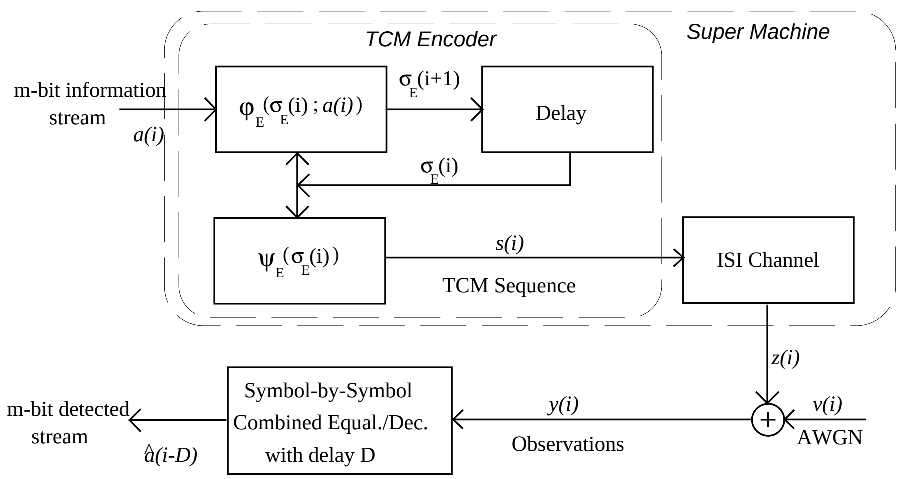

Fig. 1 - *Discrete-time low-pass version of the considered TCM-encoded ISI-impaired communication system with combined Symbol-by-Symbol equalizer/decoder of decision delay D.*

	g(0)	**g(1)**	**g(2)**	**g(3)**
Ch. A	0.58639-*j*0.29319	0.16835+*j*0.60045	-0.33713+*j*0.16856	-0.17712+*j*0.08856
Ch. B	0.853704	-0.520795	0	0

Tab.I - *List of the impulse responses of the tested ISI channels. Channel A is the same considered in [5, Sect. V].*

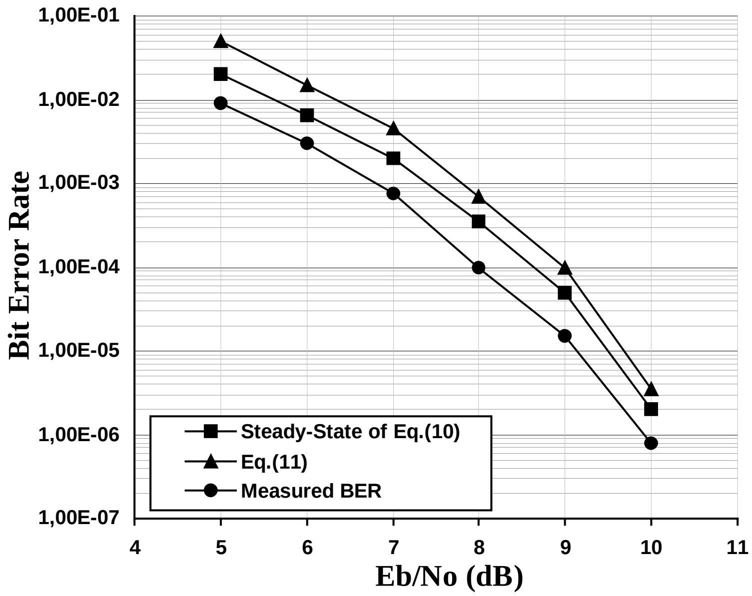

Fig. 2 - *Steady state values of the bounds of eqs.(10), (11) and measured BERs for the channel A of Tab.I.*

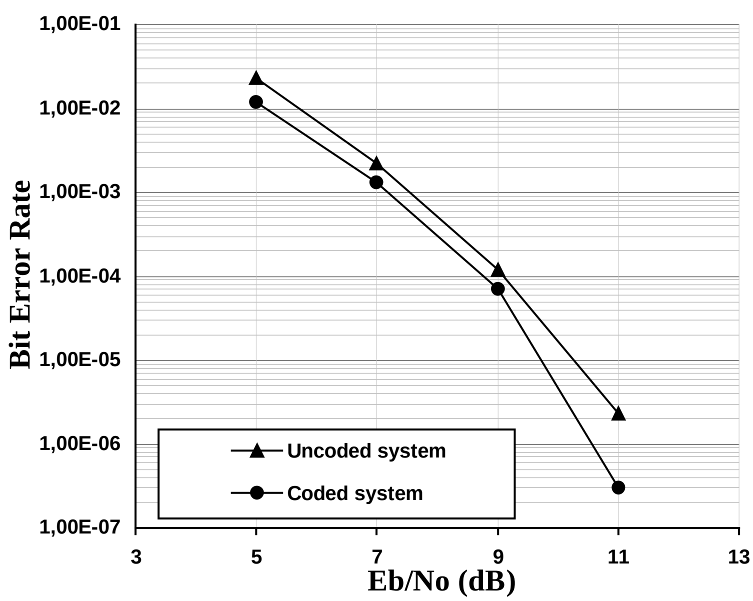

Fig. 3 - *Measured BERs for channel B. The optimized TCM encoder of eq.(13) and the corresponding BPSK-modulated SbS-equalized uncoded system are considered.*